Journal of Engineering Mathematics **48:** 219–252, 2004.

On the entry of a wedge into water: The thin wedge and an all-purpose boundary-layer equation

L.E. FRAENKEL and G. KEADY[1]
Department of Mathematical Sciences, University of Bath, BATH, BA2 7AY, United Kingdom; [1] *School of Mathematics and Statistics, University of Western Australia, Australia 6907*

Received 2 January 2002; accepted in revised form 16 May 2003

Abstract. In 1932, H. Wagner formulated the problem of the entry into water of an infinite wedge moving vertically downwards with constant speed. Among much else, Wagner noted that, in the absence of gravity, viscosity and surface tension, a similarity transformation removes time from the problem. Many other authors have considered the problem since 1932. The present paper settles a question, left open in earlier work, concerning the contact angle $\pi\beta$; this angle is shown, together with the wedge angle (or vertex angle) $2\pi\alpha$, in Figure 1(b). The question is whether the supremum $\pi\overline{\beta}$ of $\pi\beta$, over the whole set of solutions having $0 < 2\pi\alpha < \pi$, is equal to $\pi/4$ or to a smaller value. The answer is that $\overline{\beta} < 1/4$ (and the proof suggests that $1/4 - \overline{\beta}$ is not small relative to the range of β); this has long been indicated by numerical work, but (as far as we know) has not been proved rigorously until now. The paper also introduces an integral equation of boundary-layer type that allows numerical calculation *without extrapolation* of the limiting solution as $\alpha \to 0$ and of the value β_0 corresponding to $\alpha = 0$. (Such calculation is not possible with the full integral equation governing the problem.) It turns out that this same integral equation of boundary-layer type also governs the other two critical cases of the problem: $\beta \to 0$ and $\beta \to 1/4$; therefore it may be called an *all-purpose* boundary-layer equation. Numerical calculation with this equation indicates that $\beta_0 = \overline{\beta}$ and that $\beta_0 = 0.100 \pm 0.002$, which is essentially in agreement with earlier, extrapolated values.

Key words: entry of wedge into water, integral equations

1. Introduction

This paper concerns the free-boundary problem formulated by Wagner [1] of an infinite wedge moving vertically downwards with constant velocity and meeting, at time zero, the horizontal free surface of water. One wishes to describe the subsequent motion of this free boundary and of the water. Gravity, viscosity and surface tension being neglected, there is a similarity transformation that reduces the number of independent variables from three to two (but does not remove the difficulty of two nonlinear boundary conditions).

The paper supplements two others: a long paper [2] presenting, in many steps and with many estimates, a proof of the existence of solutions[1] ; and a short paper [3] that uses the existence theory to derive approximations to the flow for blunt wedges, that is, for wedge angles close to π. The reader is *not* assumed to have knowledge of either [2] or [3].

There are two important angles in the problem (Figure 1 (b)): *the wedge angle* (or vertex angle) $2\pi\alpha$ and the *contact angle* $\pi\beta$, which is that between the wedge face AB and the

[1]The proof of the existence theorem was completed (in full detail) in 1994; that the paper has been written only partly is the result of procrastination by L.E.F., who can only apologize abjectly to the readers and editor of the present paper.

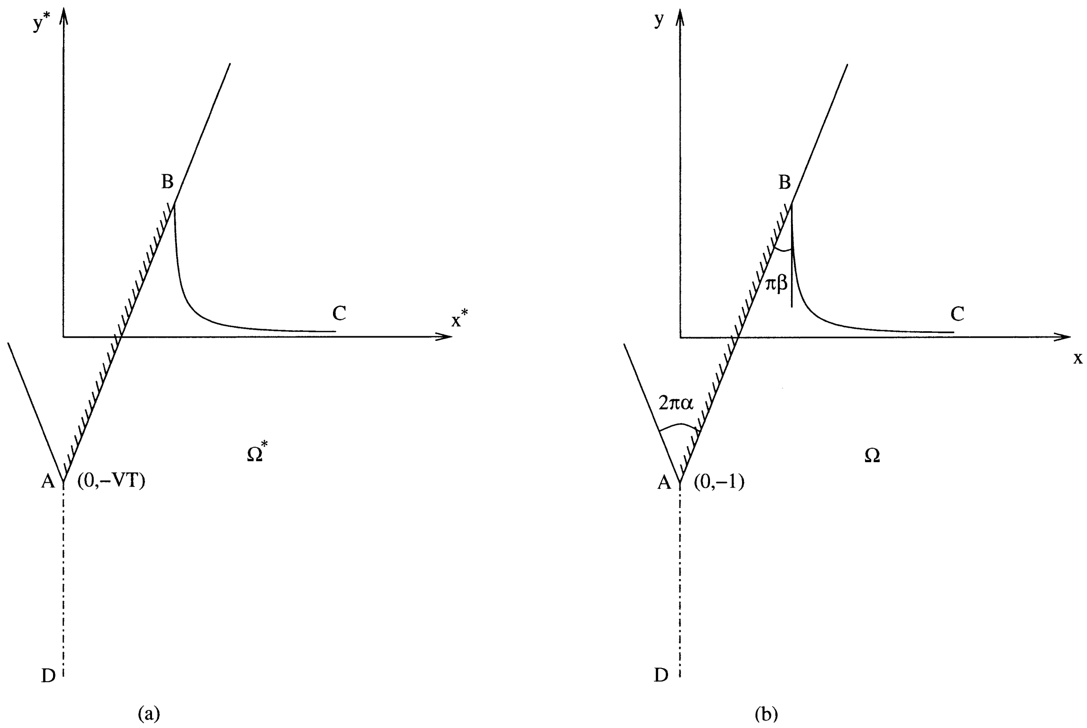

Figure 1. Notation.

tangent to the free boundary BC at the *contact point* B. The integral equation to which the problem is reduced (in all three papers, [2], [3] and the present one), and which resembles that used by Dobrovol'skaya [4] for numerical calculation, *admits solutions only if* $0 < \beta < 1/4$. This was already noted and discussed by Garabedian [5], Dobrovol'skaya [4] and Mackie [6].

That $\beta < 1/4$ is perhaps surprising: one expects that (in the absence of viscosity and surface tension) the water should become undisturbed as $\alpha \to 0$, so that $\pi\beta \to \pi/2$. The explanation is that there is a boundary-layer phenomenon near the contact point as $\alpha \to 0$: in a small region near B, of width proportional to α, disturbances are not small as $\alpha \to 0$.

Notation. Throughout the paper the statements $\alpha \to 0$, $\alpha \to 1/2$, $\beta \to 0$ and $\beta \to 1/4$ refer to *positive* values of α, $1/2 - \alpha$, β and $1/4 - \beta$, unless the contrary is explicitly stated.

The necessary condition $\beta < 1/4$ is an embarrassment if one prescribes $\alpha \in (0, 1/2)$ and seeks β as part of the solution. For this reason, and for two others to be explained at the end of Section 2.3, *we prescribe the contact angle* $\pi\beta$ *with* $0 < \beta < 1/4$ *and calculate the wedge angle* $2\pi\alpha$ *a posteriori*. The solutions established in [2] reside in the product space $(0, 1/4) \times Y$ of the interval $(0, 1/4)$ housing β and the Banach space Y housing functions h that determine the free boundary. It is proved in [2] that there is a connected set Γ of solutions (β, h) in $(0, 1/4) \times Y$ such that $\alpha(\beta, h) \to 1/2$ as $\beta \to 0$ and such that every value α in the open interval $(0, 1/2)$ occurs at least once. Let $\overline{\beta}$ denote the supremum of β over Γ; it is not known from [2] whether $\overline{\beta} = 1/4$ or $\overline{\beta} < 1/4$. Let β_0 be, loosely speaking, the first zero of $\alpha(\beta)$ as β increases from 0; more precisely, let

$$\beta_0 := \inf\{\beta_* \in (0, \overline{\beta}\,] \mid \liminf_{\beta \to \beta_*} \alpha(\beta, h) = 0\}. \tag{1.1}$$

It is not known from [2] whether $\beta_0 = \overline{\beta}$; nor whether the map $\beta \mapsto \alpha(\beta, h)$ is a continuous function or some set-valued horror.

On the other hand, reports of numerical calculation [4, 7, 8] display either $\beta(\alpha)$ or $\alpha(\beta)$ as a beautiful monotonic curve, with $\beta_0 = \overline{\beta}$ and approximately equal to 0.1; moreover, the various calculations show good agreement; see Section 6 below. But there is a difficulty even here: the value of β_0 is necessarily the result of *extrapolation* in all these calculations, because algorithms designed for the interior of the interval $(0, 1/2)$ of α, or of the interval $(0, \beta_0)$ of β, cannot deal with the gradients in the boundary layer that forms near the contact point as $\alpha \to 0$.

The contributions of the present paper are

(a) a proof that $\overline{\beta} < 1/4$, ending with a statement that is quantitative and suggests that $\overline{\beta}$ is substantially less than $1/4$;

(b) derivation of a boundary-layer approximation to the basic integral equation that allows β_0 to be calculated numerically without extrapolation;

(c) a demonstration that this boundary-layer equation governs all three critical cases of the problem, namely, $\beta \to 0,\ \beta \to \beta_0$ and $\beta \to 1/4$;

(d) a more complete picture of the flow for $\alpha \to 0$, in that the boundary-layer solution for $\beta = \beta_0$ provides an inner approximation that complements (in the sense of matched asymptotic expansions) the outer approximation derived by Mackie [9].

2. Formulation of the mathematical problem

2.1. The idealized physical problem

At times $T \leq 0$, liquid at rest occupies the half-space $\{(x^*, y^*) \in \mathbb{R}^2 \mid y^* < 0\}$. The $*$ distinguishes physical variables from the reduced variables to be introduced in Section 2.2. An infinite wedge, of vertex angle $2\pi\alpha$ $(0 < \alpha < 1/2)$, moves downwards with constant speed $V > 0$ for all time; its vertex meets the origin $(0, 0)$ at time $T = 0$. At times $T > 0$, the fluid domain is the open set $\Omega^* \subset \mathbb{R}^2$ shown in Figure 1(a) and bounded by the wedge face AB, the free boundary BC and the vertical line DA below the vertex A of the wedge.

Let (u^*, v^*) be the fluid velocity and let a^* and b^* be *material co-ordinates* such that the particle or material point labelled (a^*, b^*) for all time has position $(x^*, y^*) = (a^*, b^*)$ at time $T = 0$. Then the fluid velocity is

$$(u^*, v^*) = \left(\frac{\partial x^*}{\partial T}, \frac{\partial y^*}{\partial T}\right)\bigg|_{(a^*,b^*)\text{fixed}}$$

under the fundamental map $(a^*, b^*, T) \mapsto (x^*, y^*)$. Whether, in any given statement, u^* and v^* are to be regarded as functions of (a^*, b^*, T) or of (x^*, y^*, T) or of (x, y) or of ... will be implied by the context, throughout the paper.

The momentum equation for an inviscid fluid of constant density $\rho > 0$, subject to no extraneous force, may be written

$$\left(\frac{\partial^2 x^*}{\partial T^2}, \frac{\partial^2 y^*}{\partial T^2}\right)\bigg|_{(a^*,b^*)\text{fixed}} = -\frac{1}{\rho}\left(\frac{\partial p^*}{\partial x^*}, \frac{\partial p^*}{\partial y^*}\right), \tag{2.1}$$

where $p^*(x^*, y^*, T)$ denotes the fluid pressure. The free boundary BC is now given the complex representation

$$z^* = Z^*(a^*, T), \quad 0 < a^* < \infty, \quad T \geq 0,$$

where $z^* = x^* + \mathrm{i}y^*$ and a^* is the material co-ordinate introduced above. We seek a free-boundary function Z^* and a complex velocity field $u^* - \mathrm{i}v^*$ satisfying six conditions. The first of these is the requirement that the velocity field (u^*, v^*) is to be divergence free, because the fluid is incompressible, and irrotational, because the fluid is inviscid. Conditions (II) to (V) are self-explanatory. Equation (2.6) results from the tangential component on BC of the momentum equation (2.1): we form the scalar product of that equation and the tangential vector $(\partial X^*/\partial a^*, \partial Y^*/\partial a^*)$, where $X^* + \mathrm{i}Y^* = Z^*$.

(I) For each fixed $T > 0$, the complex velocity $u^* - \mathrm{i}v^*$ is to be holomorphic in Ω^* (as a function of z^*) and continuous on the closure $\overline{\Omega^*}$.

(II) On the wedge face AB, the normal velocity of the fluid is to be that of the wedge:

$$u^* \cos \pi\alpha - v^* \sin \pi\alpha = V \sin \pi\alpha \quad \text{on} \quad AB. \tag{2.2}$$

(III) Symmetry about the y^*-axis demands that

$$u^* = 0 \quad \text{on} \quad AD. \tag{2.3}$$

(IV) The far velocity field is to be at most that of a dipole:

$$u^* - \mathrm{i}v^* = O(|z^*|^{-2}) \quad \text{as} \quad |z^*| \to \infty. \tag{2.4}$$

(V) The free boundary BC is to be a material curve:

$$\frac{\partial Z^*}{\partial T}(a^*, T) = (u^* + \mathrm{i}v^*)\Big|_{z^* = Z^*(a^*, T)}, \quad 0 < a^* < \infty, \quad T > 0. \tag{2.5}$$

(VI) The pressure on BC is to be constant:

$$\mathcal{R}e \quad \frac{\partial \overline{Z^*}}{\partial a^*} \frac{\partial^2 Z^*}{\partial T^2} = 0, \quad 0 < a^* < \infty, \quad T > 0, \tag{2.6}$$

where the arguments a^* and T are implied, $\mathcal{R}e$ denotes the real part, and the bar denotes complex conjugation ($\overline{z^*} := x^* - \mathrm{i}y^*$).

2.2. THE SIMILARITY TRANSFORMATION AND SOME CONSEQUENCES

There is no constant characteristic length in the problem; therefore, *VT* can be used in this rôle. We make the transformation

$$z^* = VTz \quad (z = x + \mathrm{i}y,\ T > 0), \tag{2.7a}$$

$$(u^* - \mathrm{i}v^*)(z^*, T) = V(u - \mathrm{i}v)(z), \tag{2.7b}$$

$$Z^*(a^*, T) = VTZ(a), \quad \text{where} \quad a = \frac{a^*}{VT}\ . \tag{2.7c}$$

The new variables z, $u - \mathrm{i}v$, Z and a will be called *reduced* variables; the reduced fluid domain, the same for all $T > 0$ and shown in Figure 1(b), is $\Omega = \{z \mid VTz \in \Omega^*\}$. Here and elsewhere points $(x, y) \in \mathbb{R}^2$ are identified with points $z = x + \mathrm{i}y \in \mathbb{C}$.

We now seek complex-valued functions $Z \in C^1[0, \infty) \cap C^2(0, \infty)$ and $u - \mathrm{i}v \in C(\overline{\Omega})$, with $u - \mathrm{i}v$ holomorphic in Ω, such that the reduced versions of (II) to (VI) hold. The reduced forms of (II) to (IV) are obvious. The material-curve condition (V) extends to $a = 0$ by continuity (in view of our hypothesis about Z) and becomes

$$Z(a) - aZ'(a) = (u + \mathrm{i}v)|_{z=Z(a)}, \quad 0 \le a < \infty, \tag{2.8}$$

where $(.)'$ denotes differentiation. The constant-pressure condition (VI) becomes

$$\mathcal{R}e\ \overline{Z'}Z'' = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}a}|Z'|^2 = 0, \quad 0 < a < \infty, \tag{2.9}$$

where the argument a is implied.

It is known from early papers [1, 4–6] that, if these reduced forms of (I) to (VI) admit a solution Z, $u - \mathrm{i}v$ having the smoothness specified above, then there are four noteworthy consequences, as follows. (In [2] these conclusions are derived in full detail; here brief statements will suffice.)

1. *Conservation of arc length on BC*. We demand that Z^* be continuous at $(a^*, 0)$ with $a^* > 0$, so that $Z(a)/a = Z^*(a*, T)/a^* \to 1$ as $T \to 0$ and $a \to \infty$. Then (2.4) and integration of (2.8) give

$$Z(a) = a + O(a^{-2}) \quad \text{as} \quad a \to \infty, \tag{2.10a}$$

whence (2.8), as it stands, implies that

$$Z'(a) = 1 + O(a^{-3}) \quad \text{as} \quad a \to \infty. \tag{2.10b}$$

It follows from (2.9) and continuity of Z' at 0 that

$$|Z'(a)| = 1, \quad 0 \le a < \infty, \tag{2.11a}$$

equivalently that

$$\left|\frac{\partial Z^*}{\partial a^*}(a^*, T)\right| = 1, \quad 0 \le a^* < \infty, \quad T \ge 0. \tag{2.11b}$$

(For $T = 0$, $Z^*(a^*, 0) = a^*$ by the definitions of a^* and Z^*.) Integrating (2.11b) over the interval $[a_1^*, a_2^*]$, where $0 \le a_1^* < a_2^*$, we see that *the arc length between any two material points of the free boundary* (here labelled $(a_1^*, 0)$ and $(a_2^*, 0)$) *is independent of* T.

2. *The angle* $\vartheta(a)$. Equation (2.11a) allows us to write the unit tangent to BC as

$$Z'(a) = \mathrm{e}^{i\vartheta(a)}, \tag{2.12}$$

where $\vartheta(a) \to 0$ as $a \to \infty$. If the curvature $\vartheta'(a)$ is known on $(0, \infty)$, then two integrations yield $Z(a)$; in principle the whole flow is then known. It is essentially the function ϑ' that we shall pursue.

3.*The Wagner function* W. Let $U := u - \mathrm{i}v$; we define

$$W(z_0) := \int_\infty^{z_0} U'(z)^{1/2}\mathrm{d}z \quad (z_0 \in \overline{\Omega}), \tag{2.13}$$

choosing arg $U'(z)^{1/2} = 0$ for $z = \mathrm{i}y,\ y < -1$. Under hypotheses stated by Mackie [6, pp. 7–9] and satisfied by the solutions in [2], *the image of* Ω *in the* W*-plane is a triangle.* (Wagner pointed out that W maps Ω conformally onto a polygon, but presented an incorrect polygon.)

4. *Mackie's inequality*. Recall that $2\pi\alpha$ is the wedge angle and $\pi\beta$ the contact angle. *If the free boundary is convex*, which is amply the case for the solutions in [2], *then it follows from* (2.10) *and* (2.11) *that* (*for* $0 < \alpha < 1/2$)

$$\alpha + \beta < \frac{1}{2}. \tag{2.14}$$

Mackie's proof [6, p.11] is expanded and shown to be rigorous in [2], because in a later paper (Johnstone and Mackie [10]) doubt is cast on the inequality. (The later paper contains a heuristic formula for the contact angle that is contradicted by (2.14) for values of α near 1/2.)

2.3. An integral equation

The integral equation to be presented here resembles that used by Dobrovol'skaya [4] for numerical work and was suggested to McLeod and Fraenkel by her paper. However, the present integral equation differs from Dobrovol'skaya's in details of the mapping $t \mapsto z$, in its use of a logarithmic potential and in that the contact angle $\pi\beta$ is prescribed.

The set $\{t \in \mathbb{C} \mid \mathfrak{Im}\ t > 0\}$, where $\mathfrak{Im}$ denotes the imaginary part, is to be mapped conformally onto the set Ω in the z-plane by a function $\hat{z}$. The mapping is to be as in Figure 2 and $\hat{z}$ is to be continuous on $\{t \mid \mathfrak{Im}\ t \ge 0,\ t \ne 0\}$. We assume that

$$0 < \alpha < \frac{1}{2}, \quad 0 < \beta < \frac{1}{4}\ ; \tag{2.15a,b}$$

the first on physical grounds, the second for reasons that will be explained after (2.26). Let

$$q(t) := -\frac{1}{\pi}\vartheta(a(t)), \quad 0 < t \le 1, \tag{2.16}$$

where ϑ is as in (2.12) and $t \in (0, 1]$ labels points of CB in the t-plane; we have assumed existence of a decreasing function $t \mapsto a$. Then $0 < q(t) < 1$ for $0 < t \le 1$ if CB is as in

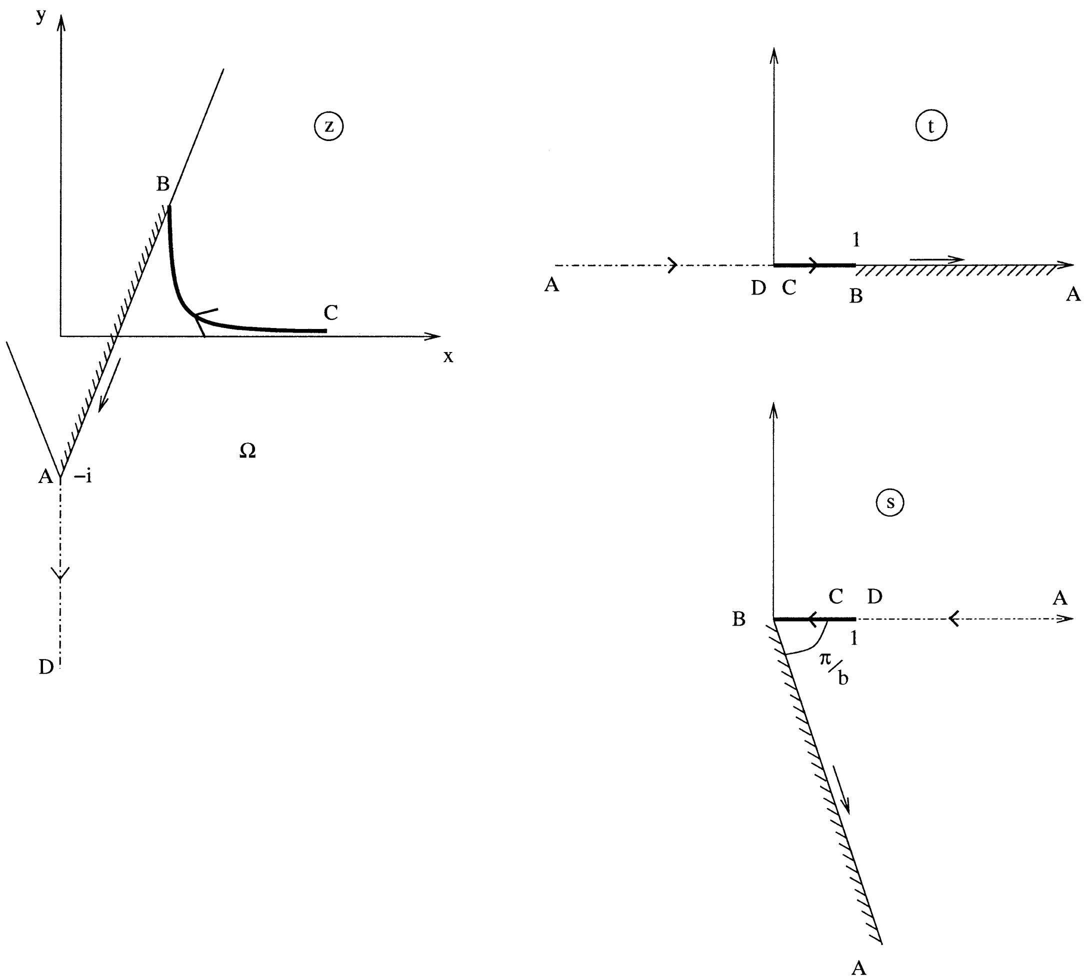

Figure 2. The domain Ω in the z-plane and its images in the t-plane and s-plane.

Figure 1; also,

$$q(0) = 0, \quad q(1) = \frac{1}{2} + \alpha - \beta\ ; \tag{2.17a,b}$$

the first by (2.10b) and the definition of q, the second because at the contact point B in the z-plane the angles $\pi q(1)$, $\pi/2 - \pi\alpha$ and $\pi\beta$ must add up to π. We demand that

$$q \in C[0,1] \cap C^1[0,1), \tag{2.18a}$$

$$0 \le q'(t) \le A(1-t)^{-1+\mu} \quad \text{for} \quad t \in [0,1), \tag{2.18b}$$

where A and μ are positive constants and $\mu < 1$.

Define the complex-valued Cauchy-Poisson integral of πq by

$$(Pq)(t) := \int_0^1 \frac{q(\tau)}{\tau - t} \mathrm{d}\tau \quad (\Im m\ t \ge 0,\ t \ne 1), \tag{2.19}$$

limiting values, as $\Im \mathrm{m}\ t \downarrow 0$, being understood when t is real and in $[0,1)$. The map $\hat{z}$, constructed by a slight extension of the principles of the Schwarz-Christoffel transformation, is defined by

$$\hat{z}(t) := -\mathrm{i} - \mathrm{i}M \int_{\infty}^{t} E(\tau)\mathrm{d}\tau \quad (\Im m\ t \geq 0,\ t \neq 0), \tag{2.20a}$$

where

$$E(t) := \mathrm{e}^{-\mathrm{i}\pi\alpha}(t-1)^{-1/2+\alpha}t^{-3/2}\exp\{-(Pq)(t)\}, \tag{2.20b}$$

arg t and arg $(t-1)$ being restricted to $[0, \pi]$. The real, positive constant M will be defined presently.

A companion formula for the complex velocity $u - \mathrm{i}v$ is obtained by first mapping $\{t \mid \Im\mathfrak{m}\ t > 0\}$ conformally onto the open triangle $W(\Omega)$ in the plane of the Wagner function W and then using the definition (2.13) of W in terms of $u - \mathrm{i}v$. The result is

$$(u - \mathrm{i}v)(t) := -\mathrm{i}N \int_0^t F(\tau)\mathrm{d}\tau \quad (\Im m\ t \geq 0), \tag{2.21a}$$

where

$$F(t) := \mathrm{e}^{i\pi(1+\alpha)}(t-1)^{-1-\alpha}\exp\{(Pq)(t)\}, \tag{2.21b}$$

arg $(t-1)$ still being restricted to $[0, \pi]$. The constants M and N are now evaluated by means of the material-curve condition (2.8) at the point $a = 0$; that is, by $\hat{z}(1) = (u + \mathrm{i}v)(1)$. The real and imaginary parts of this yield, after a contour integration of F,

$$M = \frac{\int_1^\infty |F|}{\left(\int_1^\infty |E|\right)\left(\int_{-\infty}^0 |F|\right)}, \quad N = \frac{1}{\int_{-\infty}^0 |F|}, \tag{2.22}$$

where the integrals are along the real axis.

In [2] it is proved that, *under the hypotheses* (2.15) *and* (2.18), *the formulae* (2.20) *to* (2.22) *imply that conditions* (I) *to* (IV) *and* (2.9) *are satisfied.*

It remains to incorporate the material-curve condition (2.8) for $0 < a < \infty$ (it is already satisfied for $a = 0$ by (2.22)). To this end, the logarithmic potential of $\pi q'$ is defined by

$$(Lq')(t) := \int_0^1 \log \frac{1}{|\tau - t|} \quad q'(\tau)\mathrm{d}\tau, \quad t \in \mathbb{R}; \tag{2.23}$$

condition (2.18b) ensures that Lq' *is uniformly Hölder continuous on* $\mathbb{R}$. Integration by parts and (2.17b) show that on the real axis, punctured at $t = 1$,

$$\int_0^1 \frac{q(\tau)}{\tau - t}\mathrm{d}\tau = \left(\frac{1}{2} + \alpha - \beta\right)\log|t-1| + (Lq')(t), \quad t \in \mathbb{R}\backslash\{1\}, \tag{2.24}$$

where the Cauchy principal value is used on the left-hand side when $t \in (0, 1)$. It is also proved in [2] that *Equation* (2.8) *for* $0 < a < \infty$ *is equivalent to the integral equation*

$$q'(t) = J(q')\frac{(1-t)^{-1/2-\beta}\exp\{(Lq')(t)\}}{\int_t^1 (1-\tau)^{-1+\beta}\tau^{-3/2}\exp\{-(Lq')(\tau)\}\mathrm{d}\tau} \quad (0 < t < 1), \tag{2.25}$$

where

$$J(q') := \frac{1}{\pi} \frac{\int_1^\infty (t-1)^{-1+\beta} t^{-3/2} \exp\{-(Lq')(t)\} \mathrm{d}t}{\int_1^\infty (t-1)^{-1/2-\beta} \exp\{(Lq')(t)\} \mathrm{d}t}. \tag{2.26}$$

Remarks. 1. *Absence of* α. We observe that the parameter α does not appear in (2.25) and (2.26).

2. *Necessity of the condition* $0 < \beta < 1/4$. In the denominator of (2.25), the integral exists only if $\beta > 0$. Then

$$q'(t) \sim \text{ const.} \quad (1-t)^{-1/2-2\beta} \quad \text{as} \quad t \uparrow 1, \tag{2.27}$$

so that integrability of $q'(t)$ at $t = 1$ (that is, absence of infinite spiralling of the free boundary) requires $\beta < 1/4$.

3. *Convergence of the integrals defining* $J(q')$. It is not obvious from (2.26) that the integrals there converge at infinity, but this is shown by the alternative form

$$J(q') = \frac{1}{\pi} \frac{\int_1^\infty |E|}{\int_1^\infty |F|} = \frac{1}{\pi} \frac{\int_1^\infty (t-1)^{-1/2+\alpha} t^{-3/2} \exp\{-(Pq)(t)\} \mathrm{d}t}{\int_1^\infty (t-1)^{-1-\alpha} \exp\{(Pq)(t)\} \mathrm{d}t}, \tag{2.28}$$

resulting from use of (2.24); we note that $(Pq)(t) \to 0$ as $t \to \infty$.

4. *Reasons for prescribing* β. Prescription of α would make it difficult to satisfy the condition $\beta < 1/4$. On the other hand, Mackie's inequality (2.14) ensures that $\alpha < 1/2$ when β is prescribed; the condition $\alpha > 0$ is satisfied by starting from $\beta = 0$, showing that $\lim_{\beta \to 0} \alpha(\beta) = 1/2$ [3, p.528], and then building the condition $\alpha > 0$ into the continuation procedure. Another advantage is that, in the formulae for $|E(t)|$ and $|F(t)|$ on the real axis, the logarithmic potential of $\pi q'$ accompanying β is less singular than the Hilbert transform of πq accompanying α.

5. *A monotonicity property of the integral operator.* We denote the right-hand member of (2.25) by $(Rq')(t)$ and regard R as an operator on all functions q' satisfying (2.18). Evidently $(Rq')(t) > 0$, so that R strengthens on (0, 1) the condition $q'(t) \geq 0$ postulated in (2.18) and corresponding to convexity of the free boundary. Here we record a further result of this kind.

Assume that $q \in C^2(0,1)$ and $Lq' \in C^1(0,1)$; the solutions in [2] have this additional smoothness. It is proved in [2] and again in Theorem 3.3 below that,

$$\text{if} \quad \frac{\mathrm{d}}{\mathrm{d}t} \left\{(1-t)^{1/2+2\beta} q'(t)\right\} \geq 0 \quad \text{for all} \quad t \in (0,1), \tag{2.29}$$

$$\text{then} \quad \frac{\mathrm{d}}{\mathrm{d}t} \left\{(1-t)^{1/2+2\beta} (Rq')(t)\right\} > 0 \quad \text{for all} \quad t \in (0,1). \tag{2.30}$$

It follows that, if a solution of $q' = Rq'$ satisfies (2.29) at $\beta = \beta_*$, say, and if q' and q'' vary continuously as β is varied, then strict inequality in (2.29) not only occurs at β_* but must be conserved as β varies. *In* [2] *and* [3] *solutions are constructed for small* β *that satisfy* (2.29) *with strict inequality; then* (2.30) *and a continuation procedure* (based on Leray-Schauder degree) *ensure that this property is conserved as* β *is increased.*

3. The preferred form of the integral equation

Remark 5 of Section 2.3 suggests that we should pursue not q' but the function with values $(1-t)^{1/2+2\beta}q'(t)$. Moreover, description of this function for $t \uparrow 1$ is easier with a new co-ordinate. It will also be convenient to have a logarithmic potential that vanishes at the contact point B. Accordingly, we let

$$b := \frac{2}{1-4\beta} \quad \left(0 < \beta < \frac{1}{4}, \text{ hence } 2 < b < \infty\right), \tag{3.1}$$

$$s := \{e^{-i\pi}(t-1)\}^{1/b} \quad (0 \le \arg(t-1) \le \pi), \tag{3.2a}$$

equivalently,

$$t = 1 - s^b \quad \left(-\frac{\pi}{b} \le \arg s \le 0\right); \tag{3.2b}$$

and define

$$h(s) := (1-t)^{1/2+2\beta}q'(t) \quad (s \text{ real}, 0 < s \le 1), \tag{3.3}$$

$$(Mh)(s) := (Lq')(1) - (Lq')(t) \quad (t \in \mathbb{R}). \tag{3.4}$$

Note that $q'(t)\mathrm{d}t = -bh(s)\mathrm{d}s$. The conformal map (3.2) is illustrated in Figure 2; the point $s = 0$ is the image of $t = 1$ and of the contact point in the z-plane, while $s = 1$ is the image of $t = 0$ and of infinity in the z-plane. The integral equation (2.25) becomes

$$h(s) = K(h)\frac{\exp\{-(Mh)(s)\}}{b\beta s^{-b\beta}\int_0^s \sigma^{b\beta-1}(1-\sigma^b)^{-3/2}\exp\{(Mh)(\sigma)\}\mathrm{d}\sigma}, \quad 0 < s < 1, \tag{3.5}$$

where

$$(Mh)(s) := b\int_0^1 \log\left|\frac{s^b-\sigma^b}{\sigma^b}\right| h(\sigma)\mathrm{d}\sigma, \tag{3.6}$$

$$K(h) := \frac{\beta}{\pi}\,\frac{\int_0^\infty r^{b\beta-1}(1+r^b)^{-3/2}\exp\{(M_{BA}h)(r)\}\mathrm{d}r}{\int_0^\infty r^{b/2-b\beta-1}\exp\{-(M_{BA}h)(r)\}\mathrm{d}r}, \tag{3.7}$$

$$(M_{BA}h)(r) := b\int_0^1 \log\frac{r^b+\sigma^b}{\sigma^b}\, h(\sigma)d\sigma. \tag{3.8}$$

In (3.7) and (3.8) we have set $s = re^{-i\pi/b}$. The wedge angle $2\pi\alpha$ is to be found *a posteriori* from

$$\frac{1}{2} + \alpha - \beta = q(1) = b\int_0^1 h(s)\mathrm{d}s. \tag{3.9}$$

Notation. Dependence on β will be made explicit, where this seems helpful, by the symbol $h(.;\beta)$, which need *not* denote a single function h for given β.

Following [2], we consider solutions (β, h) with h in the real Banach space Y defined by

$$Y := \{g : [0,1] \to \mathbb{R} \mid g(1) = 0, \quad Wg' \in C[0,1]\}, \tag{3.10a}$$

$$W(s) := (1-s)^{1/2}, \tag{3.10b}$$

$$\| \ g \mid Y \ \| := \max_{0 \le s \le 1} \ 2W(s) \ |g'(s)| \ , \tag{3.10c}$$

where $W(1)g'(1) := \lim_{s\uparrow 1} W(s)g'(s)$ both in this definition and in similar remarks throughout the paper.

If $h \in Y$, then $(Mh)(0) = 0$ and certainly $Mh \in C[0,1]$, so that

$$\int_0^s \sigma^{b\beta-1}(1-\sigma^b)^{-3/2} \exp\{(Mh)(\sigma)\} \mathrm{d}\sigma \sim \frac{s^{b\beta}}{b\beta} \quad \text{as} \quad s \downarrow 0\ ;$$

consequently the right-hand member of (3.5) tends to $K(h)$ as $s \downarrow 0$. Thus

$$h(0) = K(h) \quad \text{for a solution} \quad (\beta, h) \in (0, 1/4) \times Y \quad \text{of (3.5).} \tag{3.11}$$

Here is a set of *a priori* results from [2]. Proofs are included only where they are short or where they lead to the bounds (3.18) and (3.19), which will be needed in Section 4. We assume that $2 < b < \infty$.

Lemma 3.1. *If* $g \in Y$, *then* $Mg \in C^{1,1/3}[0,1]$*; that is, the derivative* $(Mg)'$ *is Hölder continuous on* $[0,1]$ *with exponent* $1/3$.

Lemma 3.2. *If* $g \in Y$ *and* $g'(s) \le 0$ *in* $(0,1)$, *which implies that* $g'(0) \le 0$ *and that* $g(s) \ge 0$ *on* $[0,1]$, *then*

$$(Mg)'(s) \ge \pi b \left(\cot \frac{\pi}{b}\right) g(s) \quad \textit{for} \quad 0 \le s \le 1. \tag{3.12}$$

The inequality is strict at points $s \in (0,1)$ *such that* $g(s) > 0$.

Proof. By the continuity of $(Mg)'$ on $[0,1]$, we need consider only points $s \in (0,1)$. Then, with $\mathcal{P}\int$ denoting the Cauchy principal value of an integral,

$$\begin{aligned}
(Mg)'(s) &= b^2 s^{b-1} \mathcal{P} \int_0^1 \frac{g(\sigma)}{s^b - \sigma^b} \,\mathrm{d}\sigma \quad (0 < s < 1) \\
&= b^2 s^{b-1} \lim_{\varepsilon \downarrow 0} \left\{ \int_0^{s-\varepsilon} \frac{g(\sigma)}{s^b - \sigma^b} \mathrm{d}\sigma - \int_{s+\varepsilon}^1 \frac{g(\sigma)}{\sigma^b - s^b} \,\mathrm{d}\sigma \right\} \\
&\ge b^2 s^{b-1} g(s) \mathcal{P} \int_0^1 \frac{1}{s^b - \sigma^b} \,\mathrm{d}\sigma \\
&= b^2 g(s) \mathcal{P} \int_0^{1/s} \frac{1}{1-\rho^b} \,\mathrm{d}\rho \\
&\ge b^2 g(s) \mathcal{P} \int_0^{\infty} \frac{1}{1-\rho^b} \,\mathrm{d}\rho \\
&= \pi b \left(\cot \frac{\pi}{b}\right) g(s)
\end{aligned}$$

by contour integration. We note that, when the interval $(0, 1/s)$ is replaced by $(0, \infty)$, there is strict inequality if $g(s) > 0$. □

Notation. With an eye on the right-hand member of (3.5), we define operators Q and T by

$$(Qg)(s) := b\beta s^{-b\beta} \int_0^s \sigma^{b\beta-1}(1-\sigma^b)^{-3/2} \exp\{(Mg)(\sigma)\}\,\mathrm{d}\sigma \quad (0 < s < 1), \tag{3.13}$$

$$(Tg)(s) := \exp\{-(Mg)(s)\}/(Qg)(s) \quad (0 < s < 1). \tag{3.14}$$

Theorem 3.3. *(Monotonicity property of T.) If $g \in Y$ and $g'(s) \le 0$ in $(0, 1)$, then $(Tg)'(s) < 0$ in $(0, 1)$.*

Proof. Setting $\sigma = s\rho$ in (3.13) and then differentiating, one obtains

$$(Qg)'(s) = b\beta \int_0^1 \rho^{b\beta-1}\{1-(s\rho)^b\}^{-3/2} \exp\{(Mg)(s\rho)\} \left[\frac{3}{2} b(s\rho)^{b-1}\{1-(s\rho)^b\}^{-1} + (Mg)'(s\rho)\right] \rho\,\mathrm{d}\rho. \tag{3.15}$$

Also,

$$(Tg)'(s) = -\frac{\exp\{-(Mg)(s)\}}{(Qg)(s)} \left\{(Mg)'(s) + \frac{(Qg)'(s)}{(Qg)(s)}\right\}. \tag{3.16}$$

Since $(Mg)'(s) \ge 0$ on $[0, 1]$ by Lemma 3.2, it follows from (3.15) that $(Qg)'(s) > 0$ in $(0, 1)$; then from (3.16) that $(Tg)'(s) < 0$ in $(0, 1)$. □

Lemma 3.4. *If $h \in Y$ and $h'(s) \le 0$ in $(0, 1)$ and (β, h) is a solution of* (3.5)*, then*

$$(1-s^b)^{3/2} \exp\{-2(Mh)(s)\} \le \frac{h(s)}{h(0)} \le \exp\{-(Mh)(s)\} \qquad \textit{for} \quad 0 \le s \le 1. \tag{3.17}$$

Proof. Let $s \in (0, 1)$. Since both $(1-\sigma^b)^{-3/2}$ and $\exp\{(Mh)(\sigma)\}$ are non-decreasing and since both equal 1 at $\sigma = 0$, we have

$$(Qh)(s) \le b\beta s^{-b\beta} \int_0^s \sigma^{b\beta-1}(1-s^b)^{-3/2} \exp\{(Mh)(s)\}\,\mathrm{d}\sigma$$

$$= (1-s^b)^{-3/2} \exp\{(Mh)(s)\},$$

and

$$(Qh)(s) \ge b\beta s^{-b\beta} \int_0^s \sigma^{b\beta-1}\,\mathrm{d}\sigma = 1.$$

These two inequalities imply those in (3.17) for $0 < s < 1$, since $K(h) = h(0)$ by (3.11); we may extend to $[0, 1]$ by the continuity of h and of Mh on $[0, 1]$. □

Lemma 3.5. *Under the hypotheses of Lemma* 3.4*,*

$$\int_0^s h \le \frac{1}{B} \log\{1 + Bh(0)s\} \qquad (0 \le s \le 1), \tag{3.18}$$

and

$$h(s) \le \frac{1}{Bs} \log\{1 + Bh(0)s\} \qquad (0 < s \le 1), \tag{3.19}$$

where $B := \pi b \cot \dfrac{\pi}{b}$.

Proof. By Lemma 3.2 and $(Mh)(0) = 0$ we have

$$(Mh)(s) \geq B \int_0^s h \qquad (0 \leq s \leq 1),$$

so that (3.17) yields

$$\frac{h(s)}{h(0)} \leq \exp\{-B \int_0^s h\}.$$

Equivalently,

$$\frac{\mathrm{d}y}{\mathrm{d}x} \leq e^{-y}, \quad \text{where} \quad x := Bh(0)s, \quad y(x) := B \int_0^s h \ .$$

Integrating this inequality, with $y(0) = 0$, we obtain (3.18). Since h is non-increasing,

$$\int_0^s h \geq sh(s),$$

which yields (3.19). □

The next theorem is the main result of [2]. Unfortunately, its proof requires far more than the preliminaries above. The first step is a demonstration that, for a solution (assumed to exist), $h(0; \beta) \to \infty$ as $\beta \to 0$ and that, after the transformation

$$s_* := h(0; \beta)s \quad (0 \leq s \leq 1), \quad h_*(s_*; \beta) := \frac{h(s; \beta)}{h(0; \beta)}, \tag{3.20}$$

any solution (β, h) with $h'(s) < 0$ in $(0, 1)$ has the limiting behaviour

$$h_*(s_*; \beta) \to \frac{1}{1 + \pi^2 s_*^2} \quad \text{as} \quad \beta \to 0 \quad \text{with } s_* \text{ fixed.} \tag{3.21}$$

(The route to (3.21) is sketched in [3, p.528]. Note that $s \to 0$ as $\beta \to 0$ with s_* fixed; there is a boundary layer near $s = 0$ as $\beta \to 0$.) It is a matter of some length to pass from (3.21) to the existence and uniqueness of exact solutions for small β. Thereafter, the continuation to larger values of β (by means of Leray- Schauder degree) also requires a long sequence of inequalities.

Theorem 3.6. *In the product space* $(0, 1/4) \times Y$ *there is a connected set* Γ *of solutions* (β, h) *as follows.*

(i) The pair (β, h) *satisfies* (3.5)*;* $h(0) = K(h)$, $h(1) = 0$ *and* $(1 - s)^{\frac{1}{2}} h'(s) < 0$ *on* $[0, 1]$ *(the limiting value being taken at* $s = 1$*).*

(ii) $Mh \in C^{1,1/3}[0, 1]$ *with* $(Mh)(0) = 0$ *and* $(Mh)'(s) > 0$ *on* $[0, 1]$.

(iii) For the wedge angle $2\pi\alpha$*, we have*

$$\alpha(\beta, h) := -\frac{1}{2} + \beta + b \int_0^1 h \ \in \ (0, \frac{1}{2} - \beta) \tag{3.22}$$

and $\alpha(\beta, h) \to 1/2$ *as* $\beta \to 0$. *Every value of* α *in the interval* $(0, 1/2)$ *occurs at least once in* Γ.

(iv) For each sufficiently small β, there is only one solution as in (i); it is described by

$$\left.\begin{aligned} h(s) &\sim \frac{\sqrt{2\beta}}{\pi}\,\frac{\sqrt{1-s^2}}{2\beta+s^2} \quad (0\le s\le 1),\\ \alpha(\beta,h) &\sim \frac{1}{2}-\sqrt{2\beta}, \end{aligned}\right\} \quad \textit{as} \quad \beta\to 0. \tag{3.23}$$

Remarks. 1. The approximation (3.23) fails to show that $h'(0) < 0$ because this derivative is of smaller order in β at $s = 0$.

2. Due to an inexcusable aberration, Theorem 2.1 in [3], which corresponds to the present Theorem 3.6, asserts more than has been proved for values of β that are not small.

4. The proof that $\overline{\beta} < 1/4$

4.1. Preliminaries

Let Γ be the connected set of solutions described in Theorem 3.6. In order to prove that

$$\overline{\beta} := \sup\{\beta \mid (\beta,h)\in\Gamma\} < \frac{1}{4}\,, \tag{4.1}$$

we shall assume the contrary: that Equation (3.5) admits solutions (β_m, h_m) as in Theorem 3.6 for a sequence (β_m) such that $\beta_m \to 1/4$. It will follow that, under this assumption, there is a subsequence such that

$$\alpha(\beta_{m(n)}, h_{m(n)})) \to -\frac{1}{4} \quad \text{as } n\to\infty. \tag{4.2}$$

This is a strong contradiction relative to the interval $(0, 1/2)$ in which α is assumed to be; it would seem to suggest that $\overline{\beta}$ is significantly less than $1/4$.

In view of (3.1), $b \to \infty$ as $\beta \to 1/4$. A preliminary re-scaling is necessary because the logarithmic potential, as it stands in (3.6), cannot tend to a limiting function as $b \to \infty$ and because Lemma 4.2 will show that $h(0; \beta) \to \infty$ as $\beta \to 1/4$. (We noted before (3.20) that this is also the case for $\beta \to 0$.) Therefore we make the transformation

$$\xi := b^2 h(0;\beta)s \quad (0\le s\le 1), \quad \hat{h}(\xi;\beta) := \frac{h(s;\beta)}{h(0;\beta)}, \tag{4.3}$$

and define

$$L = L(b) := b^2 h(0;\beta). \tag{4.4}$$

(No confusion with the operator L in (2.23) is possible.) It follows that

$$(\mathcal{M}\hat{h})(\xi) := (Mh)(s) = \frac{1}{b}\int_0^L \log\left|\left(\frac{\xi}{\xi'}\right)^b - 1\right| \hat{h}(\xi')\,\mathrm{d}\xi', \tag{4.5}$$

and that the integral equation (3.5) becomes, since $K(h) = h(0)$,

$$\hat{h}(\xi) = \frac{\exp\{-(\mathcal{M}\hat{h})(\xi)\}}{b\beta\xi^{-b\beta}\int_0^\xi x^{b\beta-1}[1-(x/L)^b]^{-3/2}\exp\{(\mathcal{M}\hat{h})(x)\}\,\mathrm{d}x}, \quad 0<\xi<L. \tag{4.6}$$

The x in (4.6) is obviously a dummy variable for ξ, quite distinct from the x in (2.7a). Equation (3.9) becomes

$$\frac{1}{4} + \alpha + \frac{1}{2b} = \frac{1}{b}\int_0^L \hat{h}(\xi;\beta)\mathrm{d}\xi \ ; \tag{4.7}$$

we wish to show that the right-hand member of this tends to zero for some sequence as $b \to \infty$.

Assumption 4.1. *Assume that* $\overline{\beta} = 1/4$.*Thus there is a sequence of solutions* (β_m, h_m) *as in Theorem* 3.6 *such that* $\beta_m \to 1/4$ *as* $m \to \infty$. *Let* $b_m := 2/(1-4\beta_m)$ *and* $L_m := L(b_m)$. *Then there is a sequence of solutions* $(\beta_m, \hat{h}_m)$ *of* (4.6) *such that, for each* $m \in \mathbb{N} := \{1, 2, 3, \ldots\}$ *and all* $\xi \in [0, L_m]$,

$$\hat{h}_m(0) = 1, \quad \hat{h}_m(L_m) = 0 \quad \textit{and} \quad \left(1 - \frac{\xi}{L_m}\right)^{1/2} \hat{h}'_m(\xi) < 0, \tag{4.8}$$

$$(\mathcal{M}\hat{h}_m)(0) = 0 \quad \textit{and} \quad (\mathcal{M}\hat{h}_m)'(\xi) > 0, \tag{4.9}$$

$$\alpha_m \in (0, \frac{1}{2} - \beta_m), \tag{4.10}$$

where α_m *is defined by* (4.7) *with* $b = b_m$ *and* $\hat{h} = \hat{h}_m$.

Notation.

$$c_b := \frac{\pi}{b}\cot\frac{\pi}{b} = 1 - \frac{1}{3}\left(\frac{\pi}{b}\right)^2 - \frac{1}{45}\left(\frac{\pi}{b}\right)^4 - \ldots (b > 2). \tag{4.11}$$

Lemma 4.2. *With the abbreviations* $\hat{h} := \hat{h}_m$, $b := b_m$ *and* $L := L_m$, *we have for each* $m \in \mathbb{N}$ *and* $b > 2$,

$$\int_0^\xi \hat{h} \le \frac{1}{c_b}\log(1 + c_b\xi) \quad (0 \le \xi \le L), \tag{4.12}$$

$$\hat{h}(\xi) \le \frac{1}{c_b\xi}\log(1 + c_b\xi) \quad (0 < \xi \le L), \tag{4.13}$$

and

$$L > \frac{1}{c_b}\left\{\exp\left(\frac{1}{4}bc_b\right) - 1\right\} > \exp\left(\frac{\pi b}{16}\right) - 1 \quad \textit{if} \quad b \ge 4. \tag{4.14}$$

Proof. The inequalities (4.12) and (4.13) are immediate consequences of Lemma 3.5 and the transformation (4.3). To prove (4.14), we use (4.7) and (4.12):

$$\frac{1}{4} + \frac{1}{2b} = -\alpha + \frac{1}{b}\int_0^L \hat{h} \ < \ \frac{1}{bc_b} \ \log(1 + c_bL),$$

whence

$$1 + c_bL > \exp\left(\frac{1}{4}bc_b + \frac{1}{2}c_b\right). \ \square$$

Note that we are still far from proving that the right-hand member of (4.7) tends to zero for some sequence as $b \to \infty$. The lower bound for $h(s)/h(0)$ in Lemma 3.4 seems to be too coarse to be useful in this context.

4.2. THE FINAL PART OF THE PROOF

The following proposition will be proved in Section 4.3. The first step of the proof is successive application of Helly's selection theorem to the sequence $(\hat{h}_m)$ on the intervals $[0, k]$, $k \in \mathbb{N}$.

Notation. $\hat{T}$ will denote the operator, analogous to the T in (3.14), that allows the right-hand member of the integral equation (4.6) to be written $\left(\hat{T}\hat{h}\right)(\xi)$.

Proposition 4.3. *(i) Under Assumption* 4.1 *there exist both a subsequence*

$$(g_n) := (\hat{h}_{m(n)})_{n=1}^{\infty}$$

of $(\hat{h}_m)$ *and a limit function* g *such that* $g_n(\xi) \to g(\xi)$ *at each fixed* $\xi \in [0, \infty)$ *as* $n \to \infty$ *and* $b_{m(n)} \to \infty$. *Moreover,*

$$g(0) = 1, \quad g \quad \textit{is non-increasing}, \quad g(\xi) \geq 0. \tag{4.15}$$

(ii) At each fixed $\xi \in [0, \infty)$, *the logarithmic potentials and right-hand members of* (4.6) *have the behaviour (as* $n \to \infty$*)*

$$(\mathcal{M} g_n)(\xi) \to \int_0^{\xi} \log \frac{\xi}{\xi'} \quad g(\xi')\mathrm{d}\xi' =: (\mathcal{M}_\infty g)(\xi), \tag{4.16}$$

$$(\hat{T} g_n)(\xi) \to \exp\{-2(\mathcal{M}_\infty g)(\xi)\}. \tag{4.17}$$

(iii) The convergence is uniform on each compact subset of $[0, \infty)$.

(iv) The unique solution of

$$g(\xi) = \exp\{-2(\mathcal{M}_\infty g)(\xi)\} \quad \textit{with} \quad g(0) = 1 \tag{4.18a}$$

is

$$g(\xi) = (1 + \xi)^{-2} \quad (0 \leq \xi < \infty). \tag{4.18b}$$

(v) Every convergent subsequence of $(\hat{h}_m)$ *converges to* $g(\xi)$ *at each fixed* $\xi \in [0, \infty)$.

Notation. The functions g_n are now extended by

$$g_n(\xi) := 0 \quad \text{for} \quad \xi > L_{m(n)}. \tag{4.19}$$

Remark. Of course, even uniform convergence on $[0, \infty)$ would not imply equality of $\lim \int_0^\infty g_n$ and $\int_0^\infty g$. But we now have the means to find an upper bound for $\int_0^\infty g_n$.

Lemma 4.4. *For* $\delta \in (0, 1/6)$ *there is a number* $N_1(\delta)$ *such that*

$$(\mathcal{M} g_n)(\xi) \geq (1 - 3\delta) \ \log(1 + \xi) \quad \textit{for all} \quad \xi \geq 0 \quad \textit{and} \quad n > N_1(\delta). \tag{4.20}$$

Proof. (i) Fix $\delta \in (0, 1/6)$ and let $\xi_0 := (1-2\delta)/\delta$. Since $g_n \to g$ uniformly on $[0, \xi_0]$, there is a number $N_0(\delta)$ such that, for $n > N_0(\delta)$,

$$g_n(\xi) \geq (1-\delta)(1+\xi)^{-2} \quad \text{if} \quad 0 \leq \xi \leq \xi_0,$$

whence

$$G_n(\xi) := \int_0^\xi g_n \;\geq\; \begin{cases} (1-\delta)\xi/(1+\xi) & \text{if} \quad 0 \leq \xi \leq \xi_0, \\ 1-2\delta & \text{if} \quad \xi \geq \xi_0. \end{cases} \tag{4.21}$$

(ii) We decompose the logarithmic potential $\mathcal{M} g_n$ into dominant and small parts. Define

$$\kappa(\eta; b) := \begin{cases} \dfrac{1}{b} \log \dfrac{1}{1-\eta^b} & \text{if} \quad 0 \leq \eta < 1, \\ \dfrac{1}{b} \log \dfrac{1}{1-\eta^{-b}} & \text{if} \quad 1 < \eta < \infty; \end{cases} \tag{4.22}$$

then

$$\mathcal{M} g_n = \mathcal{M}_\infty g_n - \mathcal{M}_\kappa g_n\,, \tag{4.23}$$

where

$$(\mathcal{M}_\infty g_n)(\xi) := \int_0^\xi \log \frac{\xi}{\xi'} \quad g_n(\xi')\mathrm{d}\xi' \;=\; \int_0^\xi \frac{G_n(\xi')}{\xi'}\, \mathrm{d}\xi', \tag{4.24}$$

$$(\mathcal{M}_\kappa g_n)(\xi) := \int_0^\infty \kappa\left(\frac{\xi'}{\xi}; b\right) \quad g_n(\xi')d\xi'\,, \tag{4.25}$$

and where $b = b_{m(n)}$. By (4.21) and (4.24),

$$(\mathcal{M}_\infty g_n)(\xi) \geq \begin{cases} (1-\delta)\;\log\,(1+\xi) & \text{if} \quad 0 \leq \xi \leq \xi_0, \\ (1-\delta)\;\log\,(1+\xi_0) + (1-2\delta)\log \dfrac{\xi}{\xi_0} & \text{if} \quad \xi \geq \xi_0, \end{cases}$$

and this last is

$$> (1-2\delta)\left\{\log(1+\xi) + \;\log\left(\frac{\xi}{1+\xi}\;\frac{1+\xi_0}{\xi_0}\right)\right\} \quad \geq \quad (1-2\delta)\;\log\,(1+\xi),$$

so that

$$(\mathcal{M}_\infty g_n)(\xi) \geq (1-2\delta)\log(1+\xi) \qquad \text{for} \quad n > N_0(\delta) \quad \text{and} \quad 0 \leq \xi < \infty. \tag{4.26}$$

(iii) In order to bound $\mathcal{M}_\kappa g_n$, we first observe that, for $b \geq 4$,

$$\int_0^1 \frac{\kappa(\eta; b)}{\eta}\,\mathrm{d}\eta + \int_1^\infty \kappa(\eta; b)\mathrm{d}\eta \;=\; \frac{1}{b^2}\int_0^1 \log\frac{1}{1-y} \quad (y^{-1} + y^{-1-1/b})\mathrm{d}y \;\leq\; \frac{A_0}{b^2},$$

where

$$A_0 := \int_0^1 \log\,\frac{1}{1-y} \quad (y^{-1} + y^{-5/4})\mathrm{d}y.$$

By (4.13) and (4.19),

$$0 \le g_n(\xi) \le \frac{4\log(1+\xi)}{\pi\xi} \quad (0 < \xi < \infty, \quad b = b_{m(n)} \ge 4).$$

This upper bound is a decreasing function of ξ, so that

$$\begin{aligned}
(\mathcal{M}_\kappa g_n)(\xi) &= \xi \int_0^\infty \kappa(\eta; b)\; g_n(\xi\eta)\mathrm{d}\eta \\
&\le \frac{4}{\pi}\xi \left\{ \int_0^1 \kappa(\eta; b)\frac{\log(1+\xi)}{\xi\eta}\mathrm{d}\eta + \int_1^\infty \kappa(\eta; b)\frac{\log(1+\xi)}{\xi}\,\mathrm{d}\eta \right\} \\
&\le \frac{4A_0}{\pi b^2}\,\log(1+\xi) \qquad (b = b_{m(n)} \ge 4) \\
&< \delta\,\log(1+\xi)
\end{aligned} \tag{4.27}$$

if n is sufficiently large, say $n > N_1(\delta) \ge N_0(\delta)$. The result (4.20) now follows from (4.23), (4.26) and (4.27). □

Lemma 4.5. *For $\delta \in (0, 1/6)$ there is a number $N_2(\delta)$ such that*

$$\int_0^\infty g_n < \frac{1+\delta}{1-6\delta} \qquad \textit{for} \quad n > N_2(\delta). \tag{4.28}$$

Proof. Fix δ at the value used in Lemma 4.4. With $\beta = \beta_{m(n)},\ b = b_{m(n)},\ L = L_{m(n)}$ and $0 < \xi < L$, also with $n > N_1(\delta)$,

$$\begin{aligned}
\hat{Q}_n(\xi) &:= b\beta\xi^{-b\beta}\int_0^\xi x^{b\beta-1}[1-(x/L)^b]^{-3/2}\ \exp\{(\mathcal{M}g_n)(x)\}dx \\
&> b\beta\xi^{-b\beta}\int_0^\xi x^{b\beta-1}(1+x)^{1-3\delta}\,dx \\
&> b\beta\xi^{-b\beta}(1+\xi)^{-3\delta}\int_0^\xi x^{b\beta-1}\,(1+x)\,dx \\
&= (1+\xi)^{-3\delta}\left(1+\frac{b\beta}{b\beta+1}\xi\right) \\
&> \frac{b\beta}{b\beta+1}(1+\xi)^{1-3\delta}.
\end{aligned}$$

We choose n so large that $(b\beta)^{-1} < \delta$, say $n > N_2(\delta) \ge N_1(\delta)$. Then

$$g_n(\xi) = \exp\{-(\mathcal{M}g_n)(\xi)\}\big/\hat{Q}_n(\xi) < (1+\delta)(1+\xi)^{-2+6\delta} \tag{4.29}$$

for $0 < \xi < \infty$, since $g_n(\xi) = 0$ for $\xi \ge L_{m(n)}$. This inequality implies (4.28). □

Theorem 4.6. *The subsequence (g_n) of $(\hat{h}_m)$ has the property that*

$$\alpha(\beta_{m(n)}, g_n) \to -\frac{1}{4} \quad \textit{as} \quad n \to \infty, \tag{4.30}$$

which contradicts (4.10). *Hence* $\overline{\beta} < 1/4$.

Proof. The result (4.30) is an immediate consequence of (4.7) and (4.28). □

4.3. PROOF OF PROPOSITION 4.3

Lemma 4.7. *Under Assumption* 4.1 *there exist a subsequence* $(g_n) := (\hat{h}_{m(n)})$ *and a limit function* g *such that* $g_n(\xi) \to g(\xi)$ *at each fixed* $\xi \in [0,\infty)$ *as* $n \to \infty$ *and such that* g *satisfies the conditions in* (4.15).

Proof. We apply Helly's selection theorem [11, p.342] successively to intervals $[0,k]$, $k \in \mathbb{N}$. The subsequence for $[0,1]$ is extracted from $(\hat{h}_m)$; that for $[0,k+1]$ is extracted from that for $[0,k]$. Finally we diagonalize. The properties of g in (4.15) are inherited from those of each g_n. □

Notation. The following definitions will be useful.

$$K(\eta;b) := \frac{1}{b}\,\log\,\left|\eta^{-b}-1\right|, \qquad \eta \in (0,\infty)\backslash\{1\}, \tag{4.31}$$

$$K_\infty(\eta) := \begin{cases} \log(1/\eta) & \text{if } 0<\eta\le 1, \\ 0 & \text{if } \eta>1, \end{cases} \qquad \kappa(\eta;b) := \begin{cases} \dfrac{1}{b}\,\log\,\dfrac{1}{1-\eta^b} & \text{if} \quad 0\le\eta<1, \\ \dfrac{1}{b}\,\log\,\dfrac{1}{1-\eta^{-b}} & \text{if} \quad \eta>1, \end{cases} \tag{4.32}$$

$$\overline{K}(\eta) := \begin{cases} \log\,(1/\eta) & \text{if} \quad 0<\eta\le 2^{-1/4}, \\ \log(1/\eta)+\kappa(\eta;4) & \text{if} \quad 2^{-1/4}<\eta<1, \\ \kappa(\eta;4) & \text{if} \quad \eta>1. \end{cases} \tag{4.33}$$

We observe that, with $g_n(\xi)=0$ for $\xi > L_{m(n)}$ and with $b=b_{m(n)}$,

$$\begin{aligned} (\mathcal{M}g_n)(\xi) &= \int_0^\infty K\left(\frac{\xi'}{\xi};b\right)\,g_n(\xi')\mathrm{d}\xi' \qquad (0<\xi<\infty) \\ &= \xi\int_0^\infty K(\eta;b)\,\,g_n(\xi\eta)\,\mathrm{d}\eta, \end{aligned} \tag{4.34}$$

and that

$$K(\eta;b) = K_\infty(\eta)-\kappa(\eta;b) \qquad \text{on} \quad (0,\infty)\backslash\{1\}, \tag{4.35}$$

$$|K(\eta;b)| \;\le\; \overline{K}(\eta) \qquad \text{on} \quad (0,\infty)\backslash\{1\}, \quad \text{if} \quad b\ge 4. \tag{4.36}$$

As in (4.24) and (4.25), $\mathcal{M}_\infty$ and $\mathcal{M}_\kappa$ are the integral operators with kernels $K_\infty(\xi'/\xi)$ and $\kappa(\xi'/\xi;b)$, respectively. The dependence on b of $\mathcal{M}$ and $\mathcal{M}_\kappa$ will be left implicit.

Lemma 4.8. $(\mathcal{M}g_n)(\xi) \to (\mathcal{M}_\infty g)(\xi)$ *at each fixed* $\xi \in [0,\infty)$ *as* $n \to \infty$.

Proof. We apply the Lebesgue dominated convergence theorem to the form (4.34) of $\mathcal{M}g_n$, assuming that $b=b_{m(n)} \ge 4$. Since $0 \le g_n(\xi\eta) \le 1$, we have

$$|K(\eta;b)g_n(\xi\eta)| \;\le\; \overline{K}(\eta) \quad \text{for} \quad \eta \in (0,\infty)\backslash\{1\},$$

and the dominant function $\overline{K} \in L_1(0,\infty)$. There is convergence almost everywhere because

$$\begin{aligned} \mid K(\eta;b)g_n(\xi\eta) - K_\infty(\eta)g(\xi\eta) \mid \quad &= \quad |\{K(\eta;b) - K_\infty(\eta)\}g_n(\xi\eta) + K_\infty(\eta)\{g_n(\xi\eta) \\ &\qquad -g(\xi\eta)\}| \\ &\le \quad \kappa(\eta;b) + K_\infty(\eta) \mid g_n(\xi\eta) - g(\xi\eta) \mid, \end{aligned}$$

which tends to zero as $n \to \infty$ with η fixed in $(0,\infty)\backslash\{1\}$. □

Lemma 4.9. *The set*

$$E := \{ \mathcal{M}f \mid 4 \le b < \infty, \quad f \in L_\infty(0,\infty), \quad \| \ f \mid L_\infty(0,\infty) \| \le \ 1\} \tag{4.37}$$

of logarithmic potentials is relatively compact in the Banach space $C[0,k]$ for fixed $k \in \mathbb{N}$.

Proof. (i) In view of the Arzelà-Ascoli theorem, we must prove that E is bounded in $C[0,k]$ and is equicontinuous on $[0,k]$. The bound will be, once continuity of $\mathcal{M}f$ is established,

$$\| \ \mathcal{M}f \mid C[0,k] \| := \max_{0\le\xi\le k} \mid (\mathcal{M}f)(\xi) \mid \ \le \ k \ \| \ \overline{K} \mid L_1(0,\infty) \ \| \ .$$

(ii) In order to prove equicontinuity, we introduce a large number N, to be chosen presently, and define

$$\begin{aligned} (\mathcal{M}f)_1(\xi) &:= \textstyle\int_N^\infty \ K\left(\dfrac{x}{\xi};b\right) \ f(x)\mathrm{d}x, \\ (\mathcal{M}f)_2(\xi) &:= \dfrac{1}{b}\textstyle\int_0^N \ \log \ \mid \xi - x \mid \ f(x)\mathrm{d}x, \\ (\mathcal{M}f)_3(\xi) &:= \dfrac{1}{b}\textstyle\int_0^N \ \log \dfrac{\xi^b - x^b}{x^b(\xi - x)} \quad f(x)\mathrm{d}x. \end{aligned}$$

Evidently $(\mathcal{M}f)(\xi)$ is the sum of these three. For $\mathcal{M}f \in E$ and $\xi \in [0,k]$,

$$\mid (\mathcal{M}f)_1(\xi) \mid = \xi \ \left| \int_{N/\xi}^\infty \ K(\eta;b) \ f(\xi\eta)d\eta \right| \le k \int_{N/k}^\infty \ \overline{K}.$$

Given $\varepsilon > 0$, we choose $N = N(\varepsilon)$ to be so large that $|(\mathcal{M}f)_1(\xi)| < \varepsilon/3$ and fix N at this value for the remainder of the proof.

(iii) The next step is to bound the derivatives $(\mathcal{M}f)_2'$ and $(\mathcal{M}f)_3'$ in the Lebesgue space $L_2(0,k)$. Now

$$(\mathcal{M}f)_2'(\xi) = \frac{1}{b} \ \mathcal{P} \int_0^N \ \frac{f(x)}{\xi - x} \ \mathrm{d}x,$$

and the Hilbert transform $L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is an isometry (Titchmarsh [12, p.123]); hence, if we allow $\xi \in \mathbb{R}$ for the moment,

$$\| \ (\mathcal{M}f)_2' \mid L_2(\mathbb{R}) \ \| \quad \le \quad \frac{\pi}{b} \ \| \ f \mid L_2(0,N) \ \| \quad \le \ \frac{\pi}{4}N^{1/2}. \tag{4.38}$$

Next,

$$(\mathcal{M}f)_3'(\xi) = \int_0^{N/\xi} \ S(\eta;b) \ f(\xi\eta)\mathrm{d}\eta \qquad \text{with} \quad S(\eta;b) := \frac{1}{b}\left(\frac{b}{1-\eta^b} - \frac{1}{1-\eta}\right),$$

and a coarse calculation shows that

$$\| \; S(.;b) \mid L_{3/2}(0,\infty) \; \| \leq \left(\frac{3}{2}\right)^{2/3} \qquad \text{for} \quad b \geq 4,$$

whence

$$\left|(\mathcal{M}f)_3'(\xi)\right| \leq \| \; S(.;b)|L_{3/2}(0,\infty) \; \| \left\{\int_0^{N/\xi} |f(\xi\eta)|^3 d\eta\right\}^{1/3} \leq \left(\frac{3}{2}\right)^{2/3}\left(\frac{N}{\xi}\right)^{1/3},$$

$$\| \; (\mathcal{M}f)_3'|L_2(0,k) \; \| \leq \left(\frac{3}{2}\right)^{2/3} N^{1/3}\left(3k^{1/3}\right)^{1/2}. \tag{4.39}$$

(iv) Let $(\mathcal{M}f)_5 := (\mathcal{M}f)_2 + (\mathcal{M}f)_3$. By (4.38) and (4.39) there is an absolute constant A_1 such that

$$\| \; (\mathcal{M}f)_5' \mid L_2(0,k) \; \| \quad \leq \quad A_1 N^{1/3}(N^{1/6} + k^{1/6}).$$

If ξ and $\xi + \delta$ (with $\delta > 0$) are in $[0,k]$ and $\varepsilon > 0$ is given, then

$$\begin{aligned} \mid (\mathcal{M}f)(\xi+\delta) - (\mathcal{M}f)(\xi) \mid \; &= \; \mid (\mathcal{M}f)_1(\xi+\delta) - (\mathcal{M}f)_1(\xi) + \textstyle\int_\xi^{\xi+\delta} (\mathcal{M}f)_5' \mid \\ &< \; \frac{2\varepsilon}{3} + \; \| \; (\mathcal{M}f)_5' \mid L_2(0,k) \; \| \; \delta^{1/2} \\ &< \; \varepsilon \end{aligned}$$

independently of f and ξ, if $\delta = \delta(\varepsilon)$ is chosen sufficiently small. □

Lemma 4.10. *There is uniform convergence of the logarithmic potentials on each compact subset of* $[0,\infty)$. *In other words, for each fixed* $k \in \mathbb{N}$,

$$\| \; \mathcal{M}_\infty g - \mathcal{M}g_n \mid C[0,k] \; \| \to 0 \quad \text{as} \quad n \to \infty.$$

Proof. Assume that the claim is false. Then there is a subsequence $(m_j) := \left(\mathcal{M}g_{n(j)}\right)$ such that, for some $k \in \mathbb{N}$ and some constant c,

$$\| \; \mathcal{M}_\infty g - m_j \mid C[0,k] \; \| \geq \; c > 0 \quad \text{for all} \quad j \in \mathbb{N}\,. \tag{4.40}$$

Since each m_j is in the relatively compact subset E defined by (4.37), there exist a subsequence $(m_{j(r)})$ and an element m_0 of $C[0,k]$ such that

$$\| \; m_0 - m_{j(r)} \mid C[0,k] \; \| \to 0 \quad \text{as} \quad r \to \infty. \tag{4.41}$$

Then $m_0 = \mathcal{M}_\infty g$ because $(\mathcal{M}g_n)(\xi) \to (\mathcal{M}_\infty g)(\xi)$ at each fixed ξ; thus (4.41) contradicts (4.40). □

Lemma 4.11. *For each fixed* $k \in \mathbb{N}$, $\hat{T}g_n \to \exp\{-2\mathcal{M}_\infty g\}$ *in* $C[0,k]$ *and* $g_n \to g$ *in* $C[0,k]$.

Proof. (i) Convergence of $(\hat{T}g_n)$ in $C[0,k]$. Let $e_n(\xi) := \exp\{(\mathcal{M}g_n)(\xi)\}$ and $e_\infty(\xi) := \exp\{(\mathcal{M}_\infty g)(\xi)\}$. By Lemmas 4.9 and 4.10, e_n and e_∞ are in $C[0,k]$ and $e_n \to e_\infty$ in $C[0,k]$. Now $\hat{T}$ is so defined that the right-hand member of (4.6) is $(\hat{T}\hat{h})(\xi)$; therefore we recall the lower bound (4.14) for $L = L_{m(n)}$ and observe that

$$\begin{aligned} \left[1-(x/L)^b\right]^{-3/2} e_n(x) - e_\infty(x) \; &= \; \{[1-(x/L)^b]^{-3/2} - 1\}e_n(x) + \{e_n(x) - e_\infty(x)\} \\ &\to 0 \quad \text{uniformly over} \quad x \in [0,k]. \end{aligned}$$

It now suffices to prove that, as $b = b_{m(n)} \to \infty$,

$$b\beta\xi^{-b\beta} \int_0^\xi x^{b\beta-1}\, e_\infty(x)\mathrm{d}x \to e_\infty(\xi),$$

equivalently, that

$$b\beta\xi^{-b\beta} \int_0^\xi x^{b\beta-1}\, \{e_\infty(\xi) - e_\infty(x)\}\mathrm{d}x \to 0,$$

uniformly over $\xi \in [0, k]$.

Let $\varepsilon > 0$ be given. Since e_∞ is uniformly continuous on $[0, k]$, there is a $\delta > 0$, independent of ξ and x in $[0, k]$, such that

$$|e_\infty(\xi) - e_\infty(x)| < \frac{\varepsilon}{2} \quad \text{for} \quad \xi - \delta < x \le \xi,$$

whence

$$\left| b\beta\xi^{-b\beta} \int_{\xi-\delta}^{\xi} x^{b\beta-1}\{e_\infty(\xi) - e_\infty(x)\}\mathrm{d}x \right| < \frac{\varepsilon}{2}.$$

Next,

$$\begin{aligned}\left| b\beta\xi^{-b\beta} \int_0^{\xi-\delta} x^{b\beta-1}\{e_\infty(\xi) - e_\infty(x)\}\mathrm{d}x \right| &\le \| e_\infty \mid C[0,k] \| \left(\frac{k-\delta}{k}\right)^{b\beta} \\ &< \frac{\varepsilon}{2}\end{aligned}$$

if $b_{m(n)}\beta_{m(n)}$ is sufficiently large.

(ii) Convergence of (g_n) in $C[0, k]$. The sequence (g_n) is a Cauchy sequence in $C[0, k]$ because $g_n = \hat{T} g_n$; also, $g_n(\xi) \to g(\xi)$. $\square$

Recall that $g_n(0) = 1$ and $g_n = \hat{T} g_n$ for each n; also, from (4.16) or (4.24), that

$$(\mathcal{M}_\infty g)(\xi) = \int_0^\xi \log \frac{\xi}{\xi'} \; g(\xi')\mathrm{d}\xi' = \int_0^\xi \frac{G(\xi')}{\xi'}\mathrm{d}\xi', \quad \text{where} \quad G(\xi) := \int_0^\xi g. \tag{4.42}$$

Therefore Lemma 4.11 implies that, for each $k \in \mathbb{N}$,

$$\left.\begin{aligned} g(\xi) = \exp\left\{-2\int_0^\xi \frac{G(\xi')}{\xi'}\,\mathrm{d}\xi'\right\} \quad &\text{for} \quad 0 < \xi \le k, \\ g \in C[0,k], \qquad &g(0) = 1. \end{aligned}\right\} \tag{4.43}$$

Let $y(\xi) := \displaystyle\int_0^\xi \frac{G(\xi')}{\xi'}\,\mathrm{d}\xi'$; then (4.43) becomes

$$\left.\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\frac{\mathrm{d}y}{\mathrm{d}\xi}\right) = \mathrm{e}^{-2y} \qquad &(0 < \xi \le k), \\ y \in C^1[0,k] \cap C^2(0,k], \quad y(0) = 0, \quad &y'(0) = 1. \end{aligned}\right\} \tag{4.44}$$

Lemma 4.12. *The unique solution of* (4.44) *is* $y(\xi) = \log(1+\xi)$, *whence* $g(\xi) = (1+\xi)^{-2}$.

Proof. That $\log(1+\xi)$ satisfies (4.44) is verified by direct substitution. The most elementary uniqueness theorem [13, p.12]) does not apply directly here because $\xi = 0$ is a singular point, but the method can be adapted to the present case.

Let $y_0(\xi) := \log(1+\xi)$ and assume that $y_0 + z$ is also a solution. Then

$$\left.\begin{array}{l} \dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi\dfrac{\mathrm{d}z}{\mathrm{d}\xi}\right) = \mathrm{e}^{-2y_0}\{\mathrm{e}^{-2z} - 1\} \qquad (0 < \xi \le k), \\ z \in C^1[0,k] \cap C^2(0,k], \quad z(0) = 0, \quad z'(0) = 0. \end{array}\right\} \tag{4.45}$$

It follows that, for $0 \le \xi \le k$,

$$z(\xi) = \int_0^\xi \frac{1}{t}\left\{\int_0^t F(s)\mathrm{d}s\right\}\mathrm{d}t, \qquad \text{where} \quad F(\xi) := \frac{\mathrm{e}^{-2z(\xi)} - 1}{(1+\xi)^2}. \tag{4.46}$$

Let $\lambda := \| z \mid C[0,k] \|$; then

$$|F(\xi)| \le A\,|z(\xi)| \le B, \qquad \text{where} \quad A := \frac{\mathrm{e}^{2\lambda} - 1}{\lambda}, \quad B := \mathrm{e}^{2\lambda} - 1.$$

We use (4.46) in the traditional manner to obtain successive inequalities.
First,

$$\left|\int_0^t F\right| \le Bt \quad \Rightarrow \quad |z(\xi)| \le B\xi.$$

Next,

$$|F(\xi)| \le AB\xi \quad \Rightarrow \left|\int_0^t F\right| \le \frac{ABt^2}{2} \quad \Rightarrow |z(\xi)| \le \frac{AB\xi^2}{4}.$$

An easy induction now shows that

$$|z(\xi)| \le \frac{A^{n-1}B\xi^n}{(n!)^2} \qquad (0 \le \xi \le k) \quad \text{for every} \quad n \in \mathbb{N},$$

which implies that $z(\xi) \equiv 0$.

Lemmas 4.7 and 4.10 to 4.12 prove (i) to (iv) of Proposition 4.3. Part (v) of the proposition is true because everything that we have said about the subsequence (g_n) is equally true for any subsequence that converges pointwise on $[0,\infty)$.

5. The boundary-layer equation

5.1. The equation and its relevance to three limiting cases

We return to Equations (4.5), (4.6) and (4.7), which are exact. At first glance it might seem that the functional $K(h)$, which weighed down the earlier form (3.5) of the integral equation, has departed; but this is not the case. The functional $L := b^2h(0)$ (the value of which is unknown *a priori*) is present in (4.5) to (4.7), and we recall that $h(0) = K(h)$ for a solution.

However, whenever it is legitimate to replace $h(0;\beta)$, and hence $L(b)$, by infinity, there is a real simplification: one then has the *boundary-layer equation*

$$\tilde{h}(\xi) = \frac{\exp\{-(\mathcal{M}\tilde{h})(\xi)\}}{b\beta\ \xi^{-b\beta} \int_0^{\xi} x^{b\beta-1} \exp\{(\mathcal{M}\tilde{h})(x)\}\mathrm{d}x}, \quad 0 < \xi < \infty, \tag{5.1}$$

where

$$(\mathcal{M}\tilde{h})(\xi) := \frac{1}{b} \int_0^{\infty} \log \left| \left(\frac{\xi}{\xi'}\right)^b - 1 \right| \tilde{h}(\xi')\ \mathrm{d}\xi'. \tag{5.2}$$

We have changed $\hat{h}$ to $\tilde{h}$ because solutions $\tilde{h}$ of (5.1) can only be limiting forms of solutions $\hat{h}$ of (4.6). The corresponding formula for the wedge angle is

$$\frac{1}{4} + \alpha + \frac{1}{2b} = \frac{1}{b} \int_0^{\infty} \tilde{h}. \tag{5.3}$$

The substitution $L = \infty$ is, in fact, legitimate for three limiting cases, as follows.

(i) $\beta \to 0$ (and hence $b \to 2$). It is a result of [2], mentioned before (3.20) and exhibited in (3.23), that $h(0;\beta) \to \infty$ as $\beta \to 0$. Here the difference between the transformations (3.20) and (4.3) is unimportant: $\xi = b^2 s_*$ and $\hat{h}(\xi) = h_*(s_*)$. The result (3.21) tells us that both re-scaled solutions of (3.5) and solutions of (5.1) have the behaviour

$$\tilde{h}(\xi;\beta) \to \frac{1}{1 + (\pi\xi/4)^2} \qquad \text{as} \quad \beta \to 0. \tag{5.4}$$

(ii) $\beta \to 1/4$ (and hence $b \to \infty$). This is the subject of the present Section 4; the factor b^2 in the transformation (4.3) is now essential. We know from Lemma 4.2 that, under Assumption 4.1,

$$h(0;\beta) \geq \frac{1}{b^2}\{\exp\left(\frac{\pi b}{16}\right) - 1\} \qquad \text{if} \quad b \geq 4,$$

so that (under Assumption 4.1) $h(0;\beta) \to \infty$ as $\beta \to 1/4$. Of course, Assumption 4.1 was ultimately shown to be false, but the proof shows that solutions of (5.1) have the behaviour

$$\tilde{h}(\xi;\beta) \to \frac{1}{(1+\xi)^2} \quad \text{as} \quad \beta \to \frac{1}{4}. \tag{5.5}$$

When one uses (5.1) to pursue the number β_0, and the limiting solution, corresponding to $\alpha = 0$, it is useful that limiting solutions of (5.1) are known explicitly both for $\beta \to 0$ and for $\beta \to 1/4$.

(iii) $\alpha \to 0$. The connected set Γ of solutions (β, h), described in Theorem 3.6, is such that every value α in the interval $(0, 1/2)$ occurs at least once. Hence there are sequences (β_n, h_n) in Γ such that

$$\alpha(\beta_n, h_n) \to 0 \quad \text{as} \quad n \to \infty\,. \tag{5.6}$$

We now show that $h_n(0) \to \infty$ for such a sequence.

Lemma 5.1. *Let* β, h *and* $\alpha = \alpha(\beta, h)$ *be as in Theorem* 3.6. *Then*

$$\frac{12}{\pi}\,\alpha\, b\, \exp\{2b^2 h(0;\beta)\} > 1\,. \tag{5.7}$$

Proof. We proceed from the identity $h(0) = K(h)$, but make the integration by parts that converts the logarithmic potential $M_{BA}h$ in (3.7) into a Hilbert transform of sorts. This is analogous to the passage from (2.26) to (2.28). Defining

$$H(s) := \int_0^s h,$$

$$(p_{BA}H)(r) := b^2 \int_0^1 \frac{r^b}{r^b + \sigma^b}\, \frac{H(\sigma)}{\sigma}\, \mathrm{d}\sigma, \tag{5.8}$$

we obtain

$$\begin{aligned} h(0) &= K(h) \\ &= \frac{\beta}{\pi}\, \frac{\int_0^\infty r^{b\beta-1}(1+r^b)^{-1+\alpha-\beta} \exp\{(p_{BA}H)(r)\}\mathrm{d}r}{\int_0^\infty r^{b/2-b\beta-1}(1+r^b)^{-1/2-\alpha+\beta} \exp\{-(p_{BA}H)(r)\}\mathrm{d}r}\,. \end{aligned} \tag{5.9}$$

It is useful to observe that the difference of the exponents $b\beta - 1$ and $b/2 - b\beta - 1$ is $2b\beta - b/2 = -1$, so that the rôle of $h(0)$ is changed by re-scaling as follows. Let

$$\lambda := h(0) \quad \text{and} \quad \rho := \lambda r, \quad \theta := \lambda\sigma, \quad \tilde{h}(\theta) := \frac{h(\sigma)}{\lambda}\,, \tag{5.10a}$$

$$\tilde{H}(\theta) := \int_0^\theta \tilde{h}(\theta')\mathrm{d}\theta' = H(\sigma). \tag{5.10b}$$

Then (5.8) and (5.9) become

$$(\tilde{p}_{BA}\tilde{H})(\rho) := (p_{BA}H)(r) = b^2 \int_0^\lambda \frac{\rho^b}{\rho^b + \theta^b}\, \frac{\tilde{H}(\theta)}{\theta}\, \mathrm{d}\theta\,, \tag{5.11}$$

$$1 = \frac{\beta}{\pi}\, \frac{\int_0^\infty \rho^{b\beta-1}[1+(\rho/\lambda)^b]^{-1+\alpha-\beta} \exp\{(\tilde{p}_{BA}\tilde{H})(\rho)\}\mathrm{d}\rho}{\int_0^\infty \rho^{b/2-b\beta-1}[1+(\rho/\lambda)^b]^{-1/2-\alpha+\beta} \exp\{-(\tilde{p}_{BA}\tilde{H})(\rho)\}\mathrm{d}\rho}\,. \tag{5.12}$$

We calculate an upper bound for the right-hand member of (5.12). Since $\tilde{h}(\theta) \le 1$, we have $\tilde{H}(\theta) \le \theta$ and

$$(\tilde{p}_{BA}\tilde{H})(\rho) \le b^2\lambda\,.$$

Both exponents of $[1 + (\rho/\lambda)^b]$ are *negative*. For the numerator in (5.12) we use

$$\begin{aligned} 1 + (\rho/\lambda)^b &\ge 1 && \text{for} \quad 0 \le \rho \le \lambda, \\ 1 + (\rho/\lambda)^b &> (\rho/\lambda)^b && \text{for} \quad \rho > \lambda. \end{aligned}$$

For the denominator we neglect the integral from $\rho = 0$ to $\rho = \lambda$ and use

$$1 + (\rho/\lambda)^b \le 2(\rho/\lambda)^b \qquad \text{for} \quad \rho \ge \lambda.$$

The important thing is that, in the denominator of (5.12), the integrand behaves like $\rho^{-1-b\alpha}$ as $\rho \to \infty$; this contributes the factor α in the result

$$1 < \frac{\alpha \, \exp(2b^2\lambda) \, 2^{1/2+\alpha-\beta}}{\pi\lambda} \left(1 + \frac{\beta}{1-\alpha}\right)$$

of our calculation. From $\int_0^1 h < h(0) = \lambda$ and (3.9) we have

$$\frac{1}{\lambda} < \frac{b}{1/2+\alpha-\beta} \quad ,$$

so that finally

$$1 < \frac{\alpha b \exp(2b^2\lambda)}{\pi} \, \frac{2^{1/2+\alpha-\beta}}{1/2+\alpha-\beta} \left(1 + \frac{\beta}{1-\alpha}\right), \tag{5.13}$$

which yields (5.7) when the inequalities $0 < \alpha < 1/2$ and $0 < \beta < 1/4$ are used. □

Theorem 5.2. *The functional $h_n(0) \to \infty$ as $n \to \infty$ whenever $\alpha(\beta_n, h_n) \to 0$.* □

Proof. Consider the inequality (5.7). Let $\overline{b} := 2/(1-4\overline{\beta})$; then Theorem 4.6 states that $b \le \overline{b} < \infty$ for all solutions. Therefore the inequality shows that $h(0; \beta) \to \infty$ as $\alpha \to 0$. □

Remarks. 1. The exact integral equation (3.5) does not admit solutions with $\alpha \le 0$, because the integral in the denominator of (5.9) would diverge at infinity, when $\alpha \le 0$, if a bounded $p_{BA}H$ were to exist. On the other hand, the boundary-layer equation (5.1) is subject to no such constraint. The restriction to positive values of α, specified in Section 1, does not apply to the boundary-layer equation.

2. We have not yet proved that (5.1) has solutions for all $\beta \in (0, 1/4)$. We have proved existence and uniqueness of solutions for values of β sufficiently close to 0 or to 1/4. Numerical calculations (Section 6) encounter no difficulty with Equation (5.1). Numerical work and asymptotic analysis of the equation for $\xi \to 0$ and for $\xi \to \infty$ both suggest that (5.4) and (5.5) give a genuine hint as to the nature of solutions. However, solutions need not decay exactly like ξ^{-2} as $\xi \to \infty$.

5.2. Inner and outer approximations for $\alpha \to 0$

(i) In this section we shall follow a path to Equation (5.1) that is of questionable rigour but that has the advantage of making Equation (5.1) one part of a more complete physical picture.

By an *outer approximation for $\alpha \to 0$* we mean one that is valid as $\alpha \to 0$ with $| z - z_B | \ge$ const. > 0 or with $| t - 1 | \ge$ const. > 0, the z-plane and t-plane being as in Figure 2. By an *inner approximation for $\alpha \to 0$* we mean one that is valid as $\alpha \to 0$ with $| z - z_B |/\alpha$ fixed or with $| t - 1 |/\alpha^2$ fixed. The reason for this scaling is as follows.

The outer approximation of the lowest order to the map $z = \hat{z}(t)$ is

$$z = e^{-i\pi/2} \, t^{-1/2} \, (t-1)^{1/2} \quad (\Im m\, t \ge 0, \quad t \ne 0, \quad \arg t \text{ and } \arg(t-1) \text{ in } [0, \pi]); \tag{5.14}$$

this corresponds to $\alpha = 0$ and to a free boundary BC along the positive x-axis in the z-plane. Recall that $\pi q(t)$ denotes the angle that the (displaced) free boundary BC in the z-plane makes with the horizontal. Mackie [9] solved a linearized problem for the velocity potential to obtain the outer approximation

$$q(t) \sim \frac{2\alpha}{\pi}\left\{\frac{1}{x} - \tan^{-1}\frac{1}{x}\right\} \qquad (0 < x < \infty) \tag{5.15a}$$

$$\sim \frac{2\alpha}{\pi}\left\{\sqrt{\frac{t}{1-t}} - \tan^{-1}\sqrt{\frac{t}{1-t}}\right\} \qquad (0 < t < 1), \tag{5.15b}$$

where $\tan^{-1}$ takes values in $(0, \pi/2)$. Although (5.15) is not valid for $x \downarrow 0$ or $t \uparrow 1$, we accept the hint given by (5.15) that in the boundary layer, where disturbances are not small, $| z - z_B |$ is proportional to α and $| t - 1 |$ is proportional to α^2. In other words, we make the guess that the transformation

$$1 - t =: \alpha^2\theta, \tag{5.16a}$$

$$q(t, \alpha) \sim q_0(\theta) \qquad \text{as} \quad \alpha \to 0 \quad \text{with} \quad \theta \quad \text{fixed}, \tag{5.16b}$$

will lead us to the lowest inner approximation. Note that $q'(t) \sim -\alpha^{-2} q_0'(\theta)$.

(ii) We seek outer and inner approximations to $q'(t)$ by means of the integral equation (2.25). Our first task is to estimate the functional $J(q')$; in the present context, (2.28) is more helpful than (2.26).

Lemma 5.3. *Let* $\gamma := 1/2 + \alpha - \beta = q(1)$ *and let* $\delta := k\alpha^2$ *for some fixed* $k > 0$. *Assume that there are (strictly) positive constants* $\overline{\alpha}$, c, A, B *and* $\underline{\beta}$ *such that for* $0 < \alpha \le \overline{\alpha}$ *we have*

$$c \le q(\tau, \alpha) \le \gamma \qquad \textit{if} \quad 1 - \delta \le \tau \le 1, \tag{5.17a}$$

$$q(\tau, \alpha) \le A\left(\frac{1-\tau}{\delta}\right)^{-1/2} + B\alpha \qquad \textit{if} \quad 0 \le \tau < 1 - \delta, \tag{5.17b}$$

and $\underline{\beta} \le \beta(\alpha) \le \overline{\beta}$. *Then there is a number* $\alpha_* \in (0, \overline{\alpha}\,]$ *such that*

$$J(q') = \frac{2\alpha}{\pi}\left\{1 + O\left(\alpha \log\frac{1}{\alpha}\right)\right\} \qquad \textit{for} \quad 0 < \alpha \le \alpha_*. \tag{5.18}$$

Remark. The term $A\,((1-\tau)/\delta)^{-1/2}$ in (5.17b) is used in anticipation of (5.25) below. However, the argument is not circular; if we replace (5.17b) by the weaker condition

$$q(\tau, \alpha) \le f\left(\frac{1-\tau}{\delta}\right) + B\alpha \quad \text{if} \quad 0 \le \tau < 1 - \delta, \quad \text{where} \quad f \in L_3(1, \infty),$$

then the only change is that in (5.18) the O-term becomes $O(\alpha^{2/3})$.

Proof of Lemma 5.3. We refer to (2.28). The main part of (5.18) results from

$$\int_1^\infty (t-1)^{-1/2}\, t^{-3/2}\, dt = 2,$$

$$\int_{1+\delta}^\infty (t-1)^{-1-\alpha}\, dt = \frac{1}{\alpha}\left\{1 + O\left(\alpha \log\frac{1}{\alpha}\right)\right\}.$$

To bound the error, we consider (2.19) and write $t =: 1 + \delta\vartheta,\ \tau =: 1 - \delta\zeta$; then

$$-(Pq)(t) = \int_0^{1/\delta} \frac{q(1-\delta\zeta)}{\zeta + \vartheta}\, d\zeta\ . \tag{5.19}$$

The inequalities (5.17) imply that

$$-(Pq)(t) \geq c \log \frac{1+\vartheta}{\vartheta} \qquad (0 < \vartheta < \infty),$$

$$-(Pq)(t) \leq \begin{cases} \gamma \log \dfrac{1+\vartheta}{\vartheta} + 2A + B\alpha \log \dfrac{1}{\delta} & \text{if} \quad 0 < \vartheta \leq 1\,, \\ \gamma \log \dfrac{1+\vartheta}{\vartheta} + A\pi\vartheta^{-1/2} + B\alpha \log \dfrac{1}{\delta} & \text{if} \quad \vartheta > 1\,. \end{cases}$$

The rest is a sequence of elementary inequalities; one example will suffice.

$$\begin{aligned} \int_1^{1+\delta} (t-1)^{-1-\alpha} \exp\{(Pq)(t)\}\, \mathrm{d}t &< \int_0^1 (\delta\vartheta)^{-1-\alpha} \vartheta^c \delta\, \mathrm{d}\vartheta \\ &= \exp\left(\alpha \log \frac{1}{k\alpha^2}\right) \frac{1}{c-\alpha} \\ &\leq \frac{2}{c} \end{aligned}$$

if $\alpha \leq \alpha_*$ and α_* is sufficiently small. □

(iii) The integral equation (2.25) may be written, by means of the integration by parts in (2.24),

$$q'(t) = J(q') \frac{(1-t)^{-1-\alpha} \exp\{(Pq)(\tau)\}}{\int_t^1 (1-\tau)^{-1/2+\alpha}\, \tau^{-3/2} \exp\{-(Pq)(\tau)\}\, \mathrm{d}\tau} \qquad (0 < t < 1). \tag{5.20}$$

Since $J(q')$ is of order α, the *outer approximation* of lowest order results from replacing α and Pq by zero elsewhere in (5.20):

$$q'(t) \sim \frac{2\alpha}{\pi} \frac{(1-t)^{-1}}{\int_t^1 (1-\tau)^{-1/2}\, \tau^{-3/2}\, \mathrm{d}\tau} = \frac{\alpha}{\pi}\, t^{1/2} (1-t)^{-3/2}, \tag{5.21a}$$

whence

$$q(t) \sim \frac{2\alpha}{\pi} \left\{ \sqrt{\frac{t}{1-t}} - \tan^{-1} \sqrt{\frac{t}{1-t}} \right\}, \tag{5.21b}$$

in agreement with Mackie's result.

(iv) For the *inner approximation* of lowest order, application of the transformation (5.16) to the definition (2.19) of Pq yields

$$(Pq)(t) \sim -\mathcal{P} \int_0^\infty \frac{1}{\theta' - \theta}\, q_0(\theta') \mathrm{d}(\theta') =: -(P_0 q_0)(\theta) \qquad (0 < \theta < \infty); \tag{5.22}$$

similarly, application of (5.16) and (5.18) to (5.20) yields

$$-q_0'(\theta) \;=\; \frac{2}{\pi}\,\frac{\theta^{-1}\,\exp\{-(P_0q_0)(\theta)\}}{\int_0^\theta (\theta')^{-1/2}\,\exp\{(P_0q_0)(\theta')\}\,\mathrm{d}\theta'} \qquad (0<\theta<\infty). \tag{5.23}$$

If this equation has a solution such that $(P_0q_0)(\theta) \to 0$ as $\theta \to \infty$, then

$$\int_0^\theta (\theta')^{-1/2}\,\exp\{(P_0q_0)(\theta')\}\,\mathrm{d}\theta' \;\sim\; 2\theta^{1/2} \quad \text{as} \quad \theta\to\infty,$$

whence

$$-q_0'(\theta) \;\sim\; \frac{1}{\pi}\,\theta^{-3/2} \quad \text{and} \quad q_0(\theta) \;\sim\; \frac{2}{\pi}\,\theta^{-1/2} \quad \text{as} \quad \theta\to\infty. \tag{5.24}$$

Equations (5.21) and (5.24) imply a *matching condition*:

$$q(t,\alpha) \;\sim\; \frac{2\alpha}{\pi}\,(1-t)^{-1/2} \tag{5.25}$$

both as $1-t \to 0$ in the outer approximation and as $\theta = (1-t)/\alpha^2 \to \infty$ in the inner approximation. (This is desirable because, if we set $1-t = \text{const.}\,\alpha^m$ with $0<m<2$ and let $\alpha\to 0$, then $1-t\to 0$ and $\theta\to\infty$.) The matching condition leads one to expect that the *composite approximation*

$$q_0(\theta) + \frac{2\alpha}{\pi}\left\{\sqrt{\frac{t}{1-t}} - \tan^{-1}\sqrt{\frac{t}{1-y}}\right\} - \frac{2\alpha}{\pi}(1-t)^{-1/2} \tag{5.26}$$

should be a lowest non-trivial approximation to $q(t,\alpha)$ for all $t\in[0,1]$. However, this is only a hope because we have not proved (a) existence of a function q_0 satisfying (5.23) and (5.24), (b) that the difference between the exact solution and (5.26) is of the expected order, say $O\left(\alpha^2\log(1/\alpha)/(\alpha+\sqrt{1-t})\right)$.

(v) We now pass from (5.23) to the form (5.1) of the boundary-layer equation. First, let

$$\beta_0 := \frac{1}{2} - q_0(0) \;=\; \frac{1}{2} + \int_0^\infty q_0'(\theta)\,\mathrm{d}\theta\,, \tag{5.27}$$

$$(L_0q_0')(\theta) := \int_0^\infty \log\frac{1}{|\theta'-\theta|}\,q_0'(\theta')\,\mathrm{d}\theta', \qquad 0\le\theta<\infty\,; \tag{5.28}$$

these are the present versions of (2.17b) and (2.23). Either by application to (5.23) of integration by parts in (5.22) or by application of (5.16) and (5.18) to (2.25) one obtains

$$-q_0'(\theta) = \frac{2}{\pi}\,\frac{\theta^{-1/2-\beta_0}\,\exp\{-(L_0q_0')(\theta)\}}{\int_0^\theta (\theta')^{-1+\beta_0}\,\exp\{(L_0q_0')(\theta')\}\,\mathrm{d}\theta'} \qquad (0<\theta<\infty). \tag{5.29}$$

Let $b_0 := 2/(1-4\beta_0)$. By analogy with (3.2) to (3.4) we introduce

$$\sigma := \theta^{1/b_0}, \quad h_0(\sigma) := -\theta^{1/2+2\beta_0}q_0'(\theta), \quad (M_0h_0)(\sigma) := (L_0q_0')(\theta) - (L_0q_0')(0), \tag{5.30}$$

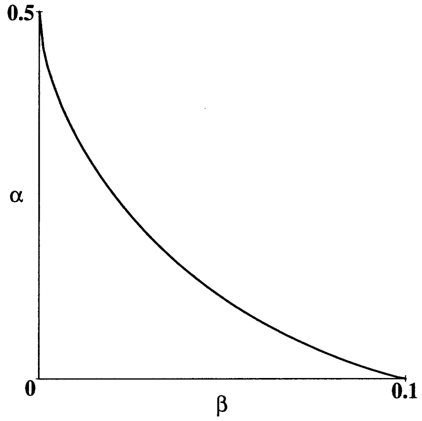

Figure 3. The graph of $\alpha(\beta)$ according to numerical solution of equation (2.25) for values of β from $1/768$ to 0.096 (Keady and Fowkes [8]).

where σ is no longer a dummy variable for s but, rather, a new co-ordinate. Because $q_0'(\theta)\mathrm{d}\theta = -b_0 h_0(\sigma)\mathrm{d}\sigma$, there results

$$(M_0 h_0)(\sigma) = b \int_0^\infty \log\left|\left(\frac{\sigma}{\sigma'}\right)^b - 1\right| \quad h_0(\sigma')\mathrm{d}\sigma', \tag{5.31}$$

$$h_0(0) = \frac{2\beta}{\pi} \exp\{-2(L_0 q_0')(0)\}, \tag{5.32}$$

$$\frac{h_0(\sigma)}{h_0(0)} = \frac{\exp\{-(M_0 h_0)(\sigma)\}}{b\beta\sigma^{-b\beta} \int_0^\sigma (\sigma')^{b\beta-1} \exp\{(M_0 h_0)(\sigma')\}\,\mathrm{d}\sigma'} \qquad (0 < \sigma < \infty), \tag{5.33}$$

in which $b = b_0$, $\quad \beta = \beta_0$ and $\beta_0 = 1/2 - b_0 \displaystyle\int_0^\infty h_0$. Finally, the transformation

$$\xi := b_0^2\, h_0(0)\, \sigma, \qquad \tilde{h}(\xi) := \frac{h_0(\sigma)}{h_0(0)} \tag{5.34}$$

yields (5.1) to (5.3) with $b = b_0$, $\beta = \beta_0$ and $\alpha = 0$.

(vi) Since Theorem 5.2 gives (5.1) to (5.3) a rigorous basis as limiting equations when $\alpha \to 0$, it is also desirable to proceed from $\tilde{h}(\xi)$ to $q_0'(\theta)$. We assume that $\tilde{h} = \tilde{h}(.;\beta_0)$ satisfies (5.1) and (5.3) with $\alpha = 0$ in (5.3); if we can use this $\tilde{h}$ to evaluate $h_0(0)$ and $(L_0 q_0')(0)$, then the passage to $q_0'(\theta)$ by way of (5.34) and (5.30) is straightforward.

Let $\lambda_0 := h_0(0)$; then (5.32) is equivalent to the equation $\lambda_0 = f(\lambda_0)$, where

$$\begin{aligned} f(\lambda) :&= \frac{2\beta_0}{\pi} \exp\left\{2\log\lambda \int_0^\infty \tilde{h} + 2\int_0^\infty \log\frac{b_0^2}{\xi}\,\tilde{h}(\xi)\,\mathrm{d}\xi\right\} \\ &= \text{const.}\,\lambda^k \qquad \text{with} \quad k := 2\int_0^\infty \tilde{h} = 1 + \frac{b_0}{2}. \end{aligned} \tag{5.35}$$

Thus $k > 1$, which ensures that f has a unique fixed point λ_0 in $(0, \infty)$. (We expect that $b_0 \approx 10/3$ so that $k \approx 8/3$.) The number $(L_0 q_0')(0)$ is now determined by (5.32).

Table 1. Some results for $\beta(\alpha)$ from Dobrovol'skaya [4] and from Zhao and Faltinsen [7]. The * labels numbers computed later for comparison with results of Keady.

α	β_D	β_{ZF}
1/2	0	0
1/3	0·011	0·009913
1/6	0·036	0·03591
1/20	0·072	0·07153
0·026	1/12	
0·0252		1/12
1/60	0·089	
10^{-2}	0·094	0·0923*
10^{-3}	0·100	0·0986*
10^{-4}	0·100	

Table 2. Some results for $\alpha(\beta)$ and $q(1;\beta)$ from Zhao and Faltinsen [7] and from Keady and Fowkes [8].

	(interpolated)		(interpolated)	
β	$\alpha\|_{ZF}$	$\alpha\|_{KF}$	$q(1)\|_{ZF}$	$q(1)\|_{KF}$
1/768	0·4453	0·4454	0·9440	0·9441
1/384	0·4202	0·4205	0·9176	0·9179
1/192	0·3834	0·3835	0·8782	0·8783
1/96	0·3286	0·3286	0·8182	0·8182
1/48	0·2487	0·2487	0·7279	0·7279
1/24	0·1419	0·1419	0·6002	0·6003
1/12	0·252	0·0253	0·4419	0·4420

6. Numerical results

Figure 3 and Tables 1 and 2 substantiate the remark in Section 1 that the numerical work of Dobrovol'skaya [4], of Zhao and Faltinsen [7] and of Keady and Fowkes [8] leads to monotonic curves for $\beta(\alpha)$ or $\alpha(\beta)$, with $\beta_0 = \overline{\beta} \approx 0 \cdot 1$, and that these various calculations show good agreement.

The numerical scheme used by Keady and Fowkes [8] to approximate solutions of (2.25), at given β, is outlined in their paper. The details of the iteration are discussed there; here we note merely that finding a good starting point for the iteration is not a difficulty. Asymptotic approximations for small β are available from [3]; one can use solutions at smaller β as good starting approximations for slightly larger β.

In Table 2 the values of β are $2^{-j}/12$ for $j = 0, 1, \ldots, 6$. The only significance of this sequence is that at the special value $\beta = 1/12$ certain checks and comparisons with exact formulae were possible. Logarithmic potentials were calculated by means of a weighted

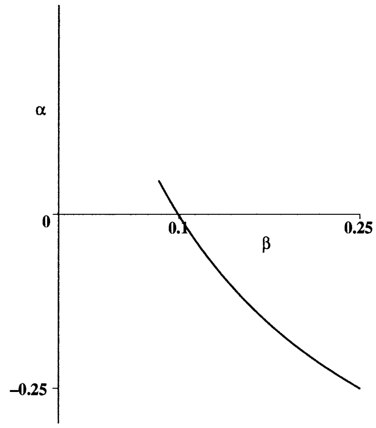

Figure 4. The graph of $\alpha(\beta)$ according to numerical solution of the boundary-layer equation (5.1) for values of β from $1/4$ to $1/12$ [14].

Table 3. Values of $\alpha(\beta)$ near $\alpha = 0$. Only the last column $(\alpha|_{K,b.l.})$ is based on the boundary-layer equation.

β	$\alpha\|_D$	$\alpha\|_{ZF}$	1997–8 $\alpha\|_K$	2002 $\alpha\|_{K,b.l.}$
0·0923		0·010	0·010	0·0204
0·094	0·010			
0·0986		0·001	0·001	0·0002
0·0995				0
0·100	0·001			

trapezoidal rule. In general, the weights were calculated by using the NAG library integration routines; for $\beta = 1/12$, the weights could be calculated exactly. Further details are available from G. Keady's web pages.

Figure 4 presents numerical results based on the boundary-layer equation (5.1); Table 3 compares such results with earlier ones near $\alpha = 0$ based on (2.25) or on variants of (2.25). As was emphasized in Section 1 and after Theorem 5.2, Equation (5.1) allows calculation on both sides of $\alpha = 0$, so that β_0 can be found by interpolation rather than extrapolation.

The numerical scheme for Equation (5.1) is described by Keady [14]. A new feature is the treatment of the logarithmic potential $\mathcal{M}\tilde{h}$ in (5.2). The decomposition $\mathcal{M} = \mathcal{M}_\infty - \mathcal{M}_\kappa$ was introduced in (4.22) to (4.25); it was evident there that $\mathcal{M}_\infty g_n$ dominates $\mathcal{M}_\kappa g_n$ when $b_{m(n)} \to \infty$. Now we add the observation that, for smooth functions f with certain decay properties,

$$(\mathcal{M}_\kappa f)(\xi) = \mu_b \xi f(\xi) + O(b^{-4}) \quad \text{as} \quad b \to \infty, \tag{6.1}$$

where

$$\mu_b := \int_0^\infty \kappa(\eta; b)\mathrm{d}\eta = 1 - \frac{\pi}{b}\cot\frac{\pi}{b} = \frac{1}{3}\left(\frac{\pi}{b}\right)^2 + \frac{1}{45}\left(\frac{\pi}{b}\right)^4 + \dots . \tag{6.2}$$

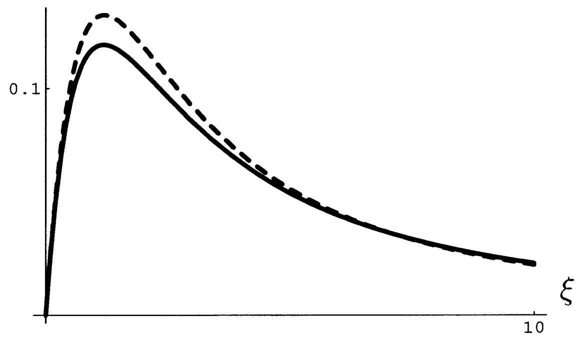

Figure 5. Graphs of $(\mathcal{M}_\kappa \tilde{h}_1)(\xi)$ (continuous curve) and $\mu_b \xi \tilde{h}_1(\xi)$ (broken curve) for $\beta = 1/10$ (hence for $b = 10/3$) and for a first approximation $\tilde{h}_1$ to the solution $\tilde{h}$ of equation (5.1).

A precise form of this observation appears below as Theorem 6.1, but for numerical treatment of (5.1) that theorem is less relevant than the following two remarks.

(i) The approximation $(\mathcal{M}_\kappa f)(\xi) \approx \mu_b\ \xi f(\xi)$ seems to be numerically better, for values of b down to 10/3, than strict analysis is likely to show; this is illustrated by Figure 5.

(ii) If $(\mathcal{M}_\kappa \tilde{h})(\xi) - \mu_b \xi \tilde{h}(\xi)$ is neglected in equation (5.1), or is regarded as known from a previous iteration, then equation (5.1) can be written as an ordinary differential equation of third order for

$$E(\xi) := \exp\{(\mathcal{M}_\infty \tilde{h})(\xi)\}, \qquad 0 \le \xi < \infty, \tag{6.3}$$

and the initial values $E(0)$, $E'(0)$, $E''(0)$ are known. This fact is basic to the numerical scheme of Keady [14].

Here is a precise form of (6.1).

Theorem 6.1. *Let $f \in C^2[0,\infty)$ with*

$$\| f^{(m)} \|_{\gamma,m} := \sup_{\xi \ge 0} (1+\xi)^{1+\gamma+m} \left| f^{(m)}(\xi) \right| < \infty \qquad \textit{for} \quad m = 0, 1, 2 \tag{6.4}$$

and for some constant $\gamma \in (0,1]$. Then there are constants $A_1(\underline{b})$ and $A_2(\underline{b})$ such that, for $b \ge \underline{b} > 3$,

$$|(\mathcal{M}_\kappa f)(\xi) - \mu_b\, \xi f(\xi)| \le \frac{1}{b^4} \left\{ A_1(\underline{b})\ \| f' \|_{\gamma,1} \frac{\xi^2}{(1+\xi)^{2+\gamma}} \right. \\ \left. + A_2(\underline{b})\ \| f'' \|_{\gamma,2} \frac{\xi^3}{(1+\xi)^{3+\gamma}} \right\}. \tag{6.5}$$

The weakness of this result, as of others in the same direction, is that $A_1(\underline{b}) \to \infty$ as $\underline{b} \downarrow 2$ and that $A_2(\underline{b}) \to \infty$ as $\underline{b} \downarrow 3$; thus (6.5) is useless for $b = 10/3$.

7. Concluding remarks

1. The main contribution of this paper has been the proof in Section 4 that, for the set of solutions established in [2], which is such that every wedge angle $2\pi\alpha$ in the open interval

$(0, \pi)$ occurs at least once, the supremum $\pi\bar{\beta}$ of the contact angle $\pi\beta$ is strictly less than $\pi/4$. In fact it was shown that, if $\pi\bar{\beta}$ *were* equal to $\pi/4$, then there would be a sequence $((\beta_n, h_n))_{n=1}^{\infty}$ of solutions for which $\alpha(\beta_n, h_n) \to -1/4$ as $n \to \infty$, contradicting strongly the fact that $0 < \alpha < 1/2$ for a solution.

2. With the result $\bar{\beta} < 1/4$ in hand, we showed relatively easily in Theorem 5.2 that there is a boundary-layer phenomenon near the contact point B as $\alpha \to 0$; more precisely, that the weighted curvature $h(0)$, of the free boundary BC at the contact point B, tends to infinity as $\alpha \to 0$. This explains why the contact angle $\pi\beta$ does not tend to $\pi/2$ as $\alpha \to 0$.

3. The boundary-layer equation (5.1), which in effect was used in [2] to construct a limiting solution for $\beta \to 0$ and $\alpha \to 1/2$, also played a significant part in Section 4 under the assumption that $\beta \to 1/4$ for a sequence of solutions. Theorem 5.2 now ensures that the exact integral equation has this same limiting form (5.1) when $\alpha \to 0$.

4. In Section 5.2 it was shown (not rigorously, but perhaps persuasively) that a transformed version of the boundary-layer equation (5.1) yields an inner approximation for $\alpha \to 0$ that appears to complement (and certainly matches) Mackie's outer approximation in [9].

5. Inevitably, more questions remain open than have been answered in [2, 3] and the present paper. Is the connected set Γ of solutions (β, h) in the product space $(0, 1/4) \times Y$ (Theorem 3.6) a set of intersecting curves? A single curve? Is the function $\beta \mapsto \alpha(\beta, h)$ monotonic, as the numerical results in Section 6 suggest? Recall that β_0 corresponds, loosely speaking, to $\alpha = 0$ and that it is defined precisely by (1.1). Does the supremum $\bar{\beta}$ equal β_0, as the results in Section 6 also suggest? Might it be that $\bar{\beta} = \beta_0 = 1/10$ exactly?

References

1. H. Wagner, Über Stoss- und Gleitvorgänge an der Oberfläche von Flüssigkeiten. *Z. Angew. Math. Mech.* 12 (1932) 193–215.
2. J. B. McLeod and L. E. Fraenkel, On the vertical entry of a wedge into water. (In preparation.)
3. L. E. Fraenkel and J. B. McLeod, Some results for the entry of a blunt wedge into water. *Phil. Trans. R. Soc. London A* 355 (1997) 523–535.
4. Z. N. Dobrovol'skaya, On some problems of similarity flow of fluid with a free surface. *J. Fluid Mech.* 36 (1969) 805–829.
5. P. R. Garabedian, Asymptotic description of a free boundary at the point of separation. *AMS Proc. Symp. Appl. Math.* 17 (1965) 111–117.
6. A. G. Mackie, The water entry problem. *Quart. J. Mech. Appl. Math.* 22 (1969) 1–17.
7. R. Zhao and O. Faltinsen, Water entry of two-dimensional bodies. *J. Fluid Mech.* 246 (1993) 593–612.
8. G. Keady and N. D. Fowkes, The vertical entry of a wedge into water: integral equations and numerical results. In: E. O. Tuck and J. A. K. Scott (eds.), *Proceedings of the Engineering Mathematics and Applications Conference.* Adelaide: Institute of Engineers,Australia (1998) pp. 277–280.
9. A. G. Mackie, A linearised theory of the water entry problem. *Quart. J. Mech. Appl. Math.* 15 (1962) 137–151.
10. E. A. Johnstone and A. G. Mackie, The use of Lagrangian coordinates in the water entry and related problems. *Proc. Camb. Phil. Soc.* 74 (1973) 529–538.
11. E. C. Titchmarsh, *Eigenfunction Expansions,* part II. Oxford: Clarendon (1958) 404 pp.
12. E. C. Titchmarsh, *Introduction to the Theory of Fourier Integrals.* Oxford: Clarendon (1948) 394 pp.
13. J. C. Burkill, *The Theory of Ordinary Differential Equations.* London: Longman (1975) 121 pp.
14. G. Keady, On a boundary layer in the problem of a wedge entering water. In: S. Wang and N. D. Fowkes (eds.), *Proceedings of the BAIL2002 Conference*. Perth: University of Western Australia (2002) pp. 153–158.